Nationaal Lucht- en Ruimtevaartlaboratorium
National Aerospace Laboratory NLR

Executive summary

An Adaptive Directional Importance Sampling method for structural reliability

Problem area
Engineering structures in general should exhibit a small probability of failure, like in the nuclear or aircraft industry. Computation of the probability of failure requires the solution of a probabilistic problem, in which uncertainty in the model parameters is taken into account by means of their distribution functions. Well known and often applied methods are the Monte-Carlo simulation and first-order reliability method. The first is very robust but inefficient and the latter is very efficient but not robust.

Description of work
In this paper a highly efficient, accurate and robust probabilistic method called adaptive directional importance sampling (ADIS) is presented. The algorithm is based on a directional simulation scheme in which the most important directions are sampled exact and the others by means of a response surface approach. These most important directions are determined by a β-sphere enclosing the most important part(s) of the limit state. The β-sphere and response surface are constantly updated during sampling with information that becomes available from the exact evaluations making the scheme adaptive.
The method has been implemented in the NLR in-house general purpose reliability analysis program RAP++.

Results and conclusions
Various widely used test problems, representing a broad range of complex limit states that can occur in practice, of which several that pose potential problems to stochastic methods in general, demonstrate the high efficiency, accuracy and robustness of the method. As such, the ADIS method is of particular interest in applications with a low probability of failure and medium number (up to about 40) of stochastic variables, for instance in aircraft and nuclear industry.

Applicability
Due to its high efficiency, accuracy and robustness, the ADIS method is particularly suited to design structural components with a low probability of failure, like required in the nuclear, aircraft and space industry.

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An Adaptive Directional Importance Sampling method for structural reliability

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Summary

In this paper an adaptive directional importance sampling (ADIS) method is presented. The algorithm is based on a directional simulation scheme in which the most important directions are sampled exact and the others by means of a response surface approach. These most important directions are determined by a β-sphere enclosing the most important part(s) of the limit state. The β-sphere and response surface are constantly updated during sampling with information that becomes available from the exact evaluations making the scheme adaptive. Various widely used test problems, representing a broad range of complex limit states that can occur in practice, of which several that pose potential problems to stochastic methods in general, demonstrate the high efficiency, accuracy and robustness of the method. As such, the ADIS method is of particular interest in applications with a low probability of failure and medium number (up to about 40) of stochastic variables, for instance in aircraft and nuclear industry.

Contents

Abbreviations

ADIS	Adaptive directional importance sampling
ARBIS	Adaptive radial-based importance sampling
COV	Coefficient of variation
DS	Directional simulation
FORM	First-order reliability method
JPDF	Joint probability density function
MCS	Monte Carlo simulation
MPP	Most probable points
OALSF	Overall limit-state function
RS	Response surface
SORM	Second-order reliability method

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1 Introduction

Methods to compute the failure probability is a basic research area in structural reliability analyses. The failure probability can be formulated as:

$$p_f = P\{G(\underline{x}) \le 0\} = \int_{G(\underline{x})\le 0} f(\underline{x})\, d\underline{x} \qquad (1)$$

where $\underline{x}$ represents the vector of stochastic variables of the reliability problem and $f(\underline{x})$ the joint probability density function (JPDF) in X-space. $G(\underline{x})$ is the failure or limit-state function, defining a safe state when $G > 0$ and a failure state when $G < 0$. The hyper-surface separating the safe from the failure domain $G = 0$ is called the limit state. The integral represents the volume of the joint probability density function located in the failure domain. Solution of the integral equation is not straightforward, due to the unknown JPDF and location of the limit state, which is only known implicitly in practical cases (contrary to the example problems of section 3), and the higher dimensionality for practical problems. Moreover, a single evaluation of the limit-state function G already can be computationally expensive, for instance solution of a finite element problem. For practical application the **efficiency** of the solution method, that is the number of deterministic analyses (G-function evaluations) required to arrive at a sufficiently accurate solution, is very important. In addition, ideally the method should be capable to solve the integral equation **accurately** for a broad range of problems, i.e. **robustness**.

In the past decades many methods have been presented to solve the integral equation, such as sampling methods based on Monte Carlo simulation (MCS) and directional simulation (DS) [1-2] and methods based on an analytical solution of the integral equation: first-order reliability method (FORM) and second-order reliability method (SORM) [3].

FORM and SORM approximate the limit state with, respectively, a first-order or an incomplete second-order function. Furthermore, the underlying solution method requires the solution of an optimization problem to find the smallest distance to the limit state. FORM, and to a lesser extent SORM, are often very efficient making these methods widely applied. In general, the accuracy of the solution is unknown, because either narrow confidence bounds cannot be obtained or they require an extra computational effort (e.g. importance sampling). Furthermore, neither method is robust in the case of a complex limit state, such as a highly nonlinear failure function, multiple design points (failure points or most probable points MPP) or a combination of failure functions (serial and parallel systems). An example of a series system having multiple design points is given in Figure 2.

On the other hand, MCS and DS are very inefficient compared with FORM and SORM, especially for small probability values. Nevertheless, convergence to the exact solution is guaranteed for an increasing number of simulations, and confidence bounds on the solution are

available in the case of a finite number of simulations. Furthermore, these methods are very robust in the sense that they can handle complex limit states.

Various methods have been presented to improve the efficiency of the two basic sampling methods (MCS and DS), for example [4-8], referred to as importance sampling techniques. The basic idea is to concentrate sampling near the most important part(s) of the limit state(s), that is points on $G(\underline{x}) = 0$ located closest to the origin in standard normal space (U-space). A widely applied approach is to shift the sampling centre from the origin to the design point. Frequently, a FORM analysis, having the mentioned disadvantages, is applied first to obtain knowledge about the design point. An alternative strategy is to gather knowledge about the failure domain, and thus location of the limit state(s), during sampling and use this knowledge to guide the sample domain towards the most important regions, e.g. [9]. This strategy is called an adaptive method.

Another strategy that drastically improves the efficiency is to first construct a response surface (RS) as a substitution of the real limit-state function [10-12]. With the RS the solution of the integral equation requires only very simple cheap function evaluations. The effort is shifted to the construction of the RS. The accuracy is completely determined by the accuracy of the RS, because even a crude MCS with many simulations can be applied. Ideally, the RS should provide a good approximation of all most important parts of the limit state. In [9] and [11] an adaptive scheme is applied that directs the RS towards a design point, which can give rise to inaccurate results in the case of multiple design points.

In [13] an adaptive radial-based importance sampling (ARBIS) method was presented by the author based on Monte Carlo simulations, in which an efficient adaptive DS scheme was presented to determine the optimal β-sphere that is excluded form the sampling domain, drastically reducing the required number of simulations compared to the Monte Carlo method. The method was demonstrated to be accurate and robust and can therefore be applied as a black box. The method lacked overall efficiency to be generally applicable, because many sample points are still located outside the failure domain.

The efficiency of the ARBIS method can be improved by application of a DS scheme instead of the MCS scheme. Beside this, the adaptive scheme to determine the optimal β-sphere can be used to distinguish between the most important parts of the limit state and the remainder of the domain. Application of a response surface for the latter domain, having a minor contribution to the probability of failure for which an approximate solution suffices, drastically increases the efficiency while maintaining accuracy. The method, being the subject of this paper, is called adaptive directional importance sampling ADIS and is very efficient, accurate and robust, demonstrated in section 3 on a broad range of problems collected from the literature, amongst them several that pose problems to stochastic methods in general. These characteristics make the method suitable to be applied in structural reliability, especially in the case of a low probability of failure.

2 Adaptive Directional Importance Sampling (ADIS)

A set of dependent non-normal stochastic variables $\underline{x}$ can always be transformed in to a set of independent standard normal variables $\underline{u}$, called the U-space, by applying appropriate transformations [14-16]. The remainder of the paper is therefore restricted to the U-space. Before presenting the ADIS method first the DS method, being the basis on which the ADIS method is founded, and related line-search method are briefly presented.

2.1 Directional Simulation

Besides Monte Carlo sampling also the directional simulation method [1-2] is frequently used. Instead of sampling random points $\underline{x}$ in the whole stochastic domain and determining whether these points lie in the failure domain or not, now random directions $\underline{\theta}$ are generated and the probability content in these directions is determined. For this purpose, the vector of random variables $\underline{x}$ is expressed in polar coordinates:

$$\underline{x} = R\underline{\theta} \tag{2}$$

in which R is the radius and $\underline{\theta}$ is a unit direction vector.
An unbiased estimator of the probability of failure is given by:

$$\hat{p}_f = \frac{1}{N}\sum_{i=1}^{N}\left(1-\chi_n^2\left(\beta_i^2\right)\right) \tag{3}$$

With this equation an estimate for the mean of the probability of failure can be obtained, by performing N simulations of the vector $\underline{\theta}$ and determining the distance to the limit state β_i in these directions. $\chi_n^2(\beta^2)$ is the cumulative chi-square distribution with n degrees of freedom. This distance is determined by means of a line-search algorithm, briefly presented in section 2.2.
An estimate of the variance on the estimator P_f is given by:

$$\hat{\sigma}_{P_f}^2 = \frac{1}{N(N-1)}\sum_{i=1}^{N}\left(p_i-\hat{p}_f\right)^2 \tag{4}$$

yielding a confidence interval for p_f of:

$$\hat{p}_f - z_{\alpha/2}\hat{\sigma}_{P_f} \le P_f \le \hat{p}_f + z_{\alpha/2}\hat{\sigma}_{P_f} \tag{5}$$

with $z_{\alpha/2}$ the standard normal variate for degree of confidence 1-α.

2.2 Line-search scheme

For each simulated direction a line-search is performed to locate the possible point on the limit state, i.e. distance β. The procedure is one-dimensional and schematically depicted in Figure 1. The G-value at the origin is determined once at the start of ADIS (point 0 in Figure 1) and is used as a scaling value as well. A linear function is fitted through this point and a first estimate (point 1) at a radius of U=4, thereby determining a second estimate of the limit-state point (point 2). Next, the G-function value is determined in this point and a quadratic fit is made resulting in an improved estimate. This procedure is repeated until the limit-state point is found having a pre-defined absolute limit-state function error tolerance. Usually the process converges in two-to-three iterations, because an approximate location is already obtained from the response surface.

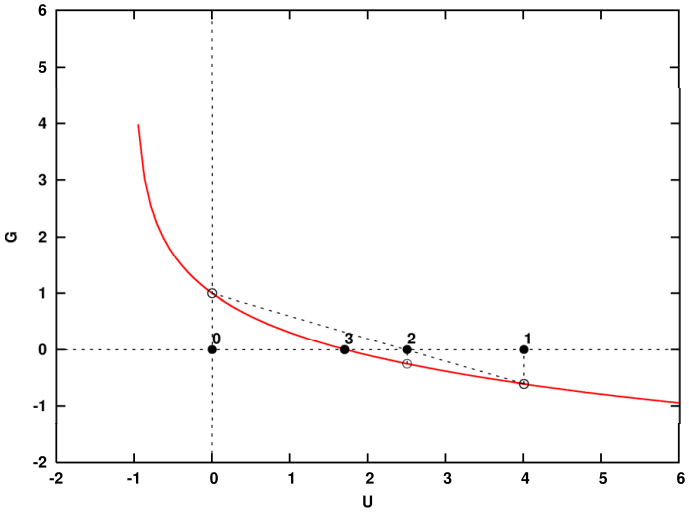

Figure 1 Line-search procedure.

2.3 Outline of the ADIS method

In order to reduce the number of simulations (improve efficiency) required for a MCS or DS more efficient simulation methods have been developed. They all are based on the fact that simulations are not performed throughout the whole stochastic domain, but only around the most important part(s) of the limit state. Generally, DS is more efficient than MCS and will therefore form the basis of the ADIS method. Only for a large number of random variables MCS requires less function evaluations than DS to converge to a solution of similar accuracy. A very important characteristic of MCS is that the solution of the integral equation converges to the exact solution as the number of simulations is increased. In general, also DS shows this behaviour. However, if part of the limit state is shielded by another part seen from the origin in U-space, for example the upper tail of problem 6 in Figure 4, then the safe area behind it will go unnoticed causing an overestimation of the failure probability (conservative). This error is

small, because these parts of the limit state are away from the most important parts and thus have only a limited contribution to the failure probability. Hence, this error is negligible certainly compared to other errors, such as a finite number of simulations (discussed in section 3) and especially compared to FORM in which the complete limit state is approximated by a first-order function.

The strategy is to gather knowledge about the failure domain and location of the limit state(s) during sampling and use this knowledge to guide the sample domain towards the most important part(s). At start no information about the limit state and its most important part(s) is available. Therefore, the method starts with the standard DS scheme performing a number of directional simulations (pre-sampling). When enough points have been sampled the obtained limit-state function information is used to construct a response surface (RS), see section 2.5 for more details. This response surface, being an approximation of the real limit-state function, is then used in the DS scheme. In addition, the obtained limit-state function information is used to determine a threshold β-sphere β_{th} , depicted in Figure 2 for the final optimal situation, enclosing the most important parts of the limit state.

The response surface is computationally very efficient, since no more expensive exact G-function evaluations are required. The downside is that the accuracy is not guaranteed by the approximate nature of the RS. Therefore, if an important direction is sampled having an (approximated) distance to the limit state β lying within the threshold β-sphere, the grey area in Figure 2 for the final optimum situation, then this direction is re-evaluated using exact G-function evaluations. The extra limit-state function information is subsequently used to improve the accuracy of the response surface and to improve the current estimate of β-sphere, making the scheme adaptive. In other words, for the most important part(s) of the limit state the probability content is determined by expensive but exact G-function evaluations and for the remainder part of the domain cheap response surface evaluations are used to approximate the corresponding small amount of probability content. In this way the method is very efficient, using only expensive G-function evaluations in the most important regions, and accurate, using exact limit state evaluations in the most important directions. The scheme is also robust since the DS method is inherently robust and because it can handle complex limit states such as a noisy failure function, highly nonlinear failure function, multiple design points and/or multiple failure functions. This robustness will be demonstrated by the numerical examples in section 3. These criteria are satisfied as long as a response surface can be found that approximates the most important part(s) of the limit state well. Finally, as a result of the exact directional simulations in these directions the resulting response surface will be a very good approximation of these areas. Even better still, most of these points will be located close to the limit state since an estimate of the limit-state point location is already available from the preceding response surface evaluation. The resulting response surface is much more accurate than by applying other

point sampling methods such as random sampling or factorial designs, which aim at a more global G-function approximation. However, away from the limit state a less accurate approximation of the limit-state function suffices, since only the sign of the gradient is important here, i.e. increasing or decreasing limit-state function in a specific direction.
In the next section, the algorithm behind ADIS will be discussed in more detail assuming that the problem is already formulated in U-space. The basic steps of the ADIS algorithm are depicted in the flow diagram of Figure 3. These steps are discussed next in more detail. Details with respect to the response surface are discussed in section 2.5.

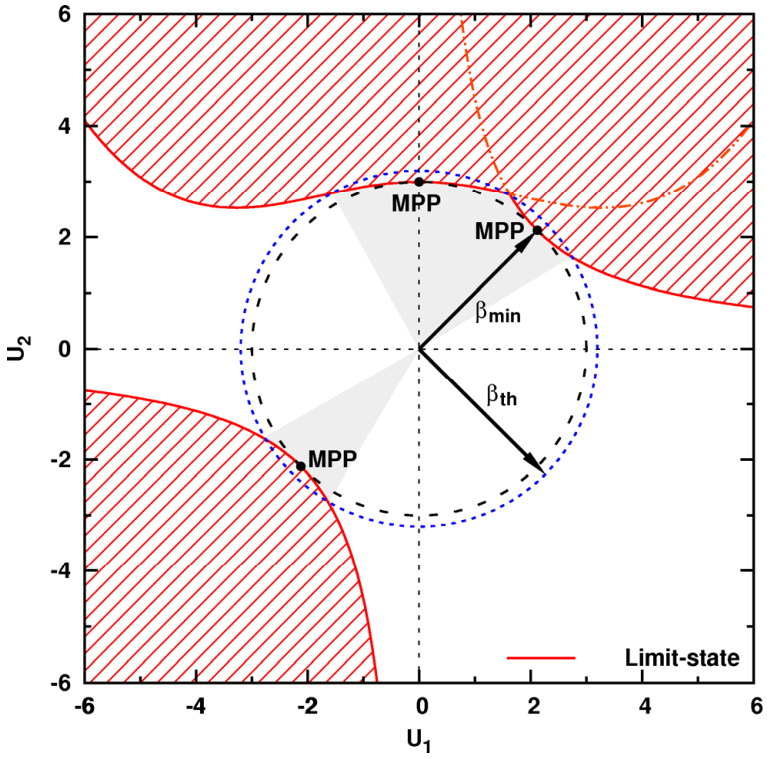

Figure 2 Schematized ADIS method.

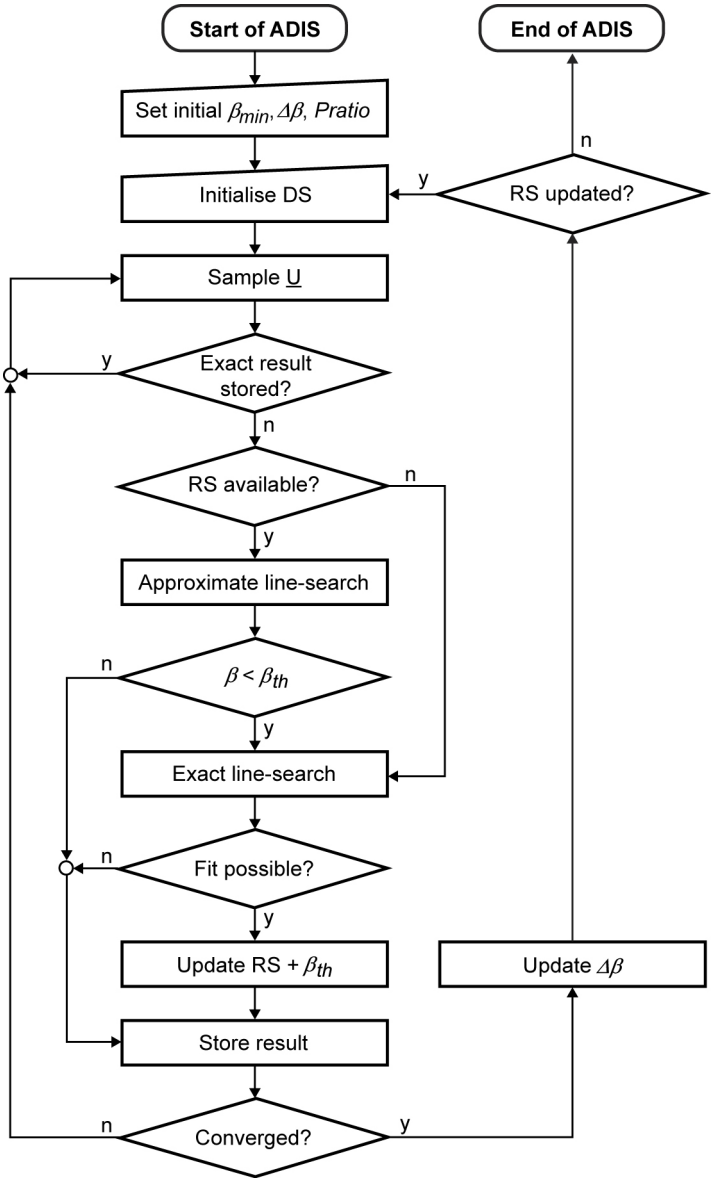

Figure 3 Basic steps of ADIS scheme.

2.4 Algorithmic details

At the start of the algorithm, initial values are set for the parameters β_{min}, $\Delta\beta$, *Pratio,* related to the optimal β-sphere discussed below. Next, the DS method is initialised, for example initialisation of the random number generator. The directional simulation is (re)started and for each sampled direction it is checked whether an exact simulation was already performed in a previous loop, in which case the next direction will be simulated. When available, the response surface will be used in a line-search to determine the approximate distance β to the limit state, otherwise, an exact line-search will be performed. The β-value is a measure for the probability content in that direction. If the approximate β-value is less than a threshold value β_{th}, then the point lies on an important part of the limit state, visualised in Figure 2 by the grey areas for the

final situation. In that case an exact line-search is performed in the same direction to determine the exact distance to the limit state. In general, only a few G-function evaluations are required to converge, because an estimate of the limit-state point is available from the approximate line-search. The corresponding points therefore all lie close to the limit state. This extra G-function information is subsequently used to improve the response surface and the threshold β-value. Due to the location of the extra points close to the limit state the resulting response surface will approximate this part of the limit state best, which is exactly the region where it should be most accurate. In all, apart from the directions sampled to generate the initial response surface, the sampling process performs almost only exact G-function evaluations in those regions that matter making the method very efficient and accurate.

After each directional simulation, convergence is checked according to the criterion discussed in section 3. At convergence, it is checked whether new exact G-function evaluations have been performed resulting in an update of the response surface and threshold β-sphere. In that case, all the approximated directions are re-evaluated against the new response surface and β-sphere. To this end, the DS algorithm is re-initialised and the whole algorithm is redone, evaluating only those directions for which no exact solution is available. Re-evaluation is only done after a converged solution has been obtained to limit the number of such restarts. If none of the approximate line-searches of a full loop is followed by an exact line-search, no improvements can be made and the algorithm ends.

The β-sphere with radius β_{th}, the dotted blue-line in Figure 2, is constructed from the current minimum distance to the limit state β_{min} with an added offset $\Delta\beta$ enclosing the most important part(s). Initially, the value of β_{min} is set to a very high value, but is updated after each exact line-search, being an adaptive process. This value converges to the real minimum β-value (see [13]). Normally, a limited number of exact line-searches suffice to confine the most important part(s) of the limit state. For the offset $\Delta\beta$ a small initial value of 0.1 is selected based on a large number of test cases. To guarantee accuracy, a new offset $\Delta\beta$ value is determined at convergence based on the maximum probability content within the approximate failure domain. To this, the ratio of the part of the failure probability determined by the approximate directions P_f^{approx} to the total failure probability P_f is defined:

$$Pratio = \frac{P_f^{approx}}{P_f} \tag{6}$$

When the solution is converged the probability content for all evaluated directions is available in a p-vector together with information whether an exact or approximate line-search was applied. The ratio can be easily computed using equation (3) and should be less than a specified

value. For the test cases a value of 0.4 resulted in accurate solutions. After sorting of the *p*-vector, a new offset $\Delta\beta$ value is determined that satisfies this criterion.

2.5 Response surface

The initial response surface is a linear or in incomplete second-order approximation of the limit-state function:

$$G(\underline{u}) = a + \sum_{i=1}^{n} b_i u_i + \sum_{i=1}^{n} c_i u_i^2 \qquad (7)$$

in which a, b_i, c_i are the unknown constants that have to be determined and n is the number of random variables. The minimum number of G-function evaluations required to make an initial fit therefore is $2n+1$. This initial response surface not necessarily is a very good approximation of the real response. At this stage, it suffices to only capture part of the global behaviour of the real response at first. This response surface is improved adaptively with extra information that comes available from additional exact directional simulations during the sampling scheme, as explained in the previous section. Hence, the response surface will converge to the one best describing the problem.

The incomplete function neglecting the cross-terms, which is frequently used by others, can lead to severe errors. Therefore, active cross-terms ($c_{ij} \neq 0$) are added when more G-function values become available during the sample process, eventually yielding a full second-order fit:

$$G(\underline{u}) = a + \sum_{i=1}^{n} b_i u_i + \sum_{i=1}^{n} \sum_{j=1}^{i} c_{ij} u_i u_j \qquad (8)$$

The number of limit-state function points required to obtain a full fit is at most:

$$N_{max} \leq 1 + n + \frac{n(n+1)}{2} \qquad (9)$$

In general, the number of random variables will be limited, since in most structural analyses the number of model parameters is limited. A problem with 25 random variables requires at maximum 351 G-function evaluations to obtain a full second-order approximation. For most problems, the number of G-function evaluations will be large enough to fit the full quadratic function, especially since many cross-terms will be zero in general.

For the RS, all limit-state function points found during exact line-searches are used. Many of these points are located near and on the limit state in the most important regions. The response surface thus approximates the global limit-state function and is best at the most important

regions in the neighbourhood of the limit state. In this way all the (expensively determined) information is located in the areas of highest interest that determine the accuracy.
Often more points are available than strictly necessary for fitting the full second-order polynomial. Therefore, the fit is obtained by a regression analysis, including all points, which improves accuracy.
For cases with multiple limit-state functions (series systems, parallel systems or combinations), such as the ones depicted in Figure 2 and Figure 4, separate response surfaces are constructed for each of them, instead of one response surface approximating the overall limit-state function (OALSF). The latter would be much less accurate, because of the non-smoothness and high nonlinearity of this function in general, which can be seen in example problems 12 and 13 of Figure 4 that have a complex limit state. At any point in the stochastic domain, the OALSF equals the minimum or maximum of the separate limit-state functions at this point for a series, respectively, parallel system (or a combination for more complex systems) and is represented by the individual response surfaces for each failure function. The OALSF value is used in the line-search to determine the location of a possible limit-state point.
Generating a response surface for each failure function does not require any extra computational effort, because at each exact G-function evaluation, blue points in Figure 4, information about each individual failure function can be obtained.
The algorithm is demonstrated for a second-order response surface, but it is stressed here that any other type of meta-model can be used as well, such as a Kriging meta-model or component meta-model [12], which may even further improve accuracy.

3 Numerical Examples

The ADIS method is applied to a set of widely used test problems obtained from the literature, representing a broad range of possible limit states that can occur in practice of which several pose potential problems to stochastic methods in general. The problems are summarized in Table 1, in which the last column gives the corresponding reference. Since these problems are used by various authors, the reference is not necessarily the first one. Because of the simple nature of the limit-state functions they can be evaluated many times, making a near exact evaluation possible by crude Monte Carlo or Directional Simulation. This near exact value is given in column 5 of Table 2. The presented ADIS methodology, however, is developed for implicit limit-state functions and therefore makes no use whatsoever of any explicit limit-state information available in the provided test problems!

Table 1 Limit-state function descriptions.

Case	Limit-state function(s)	Stochastic variables	Description	P_f	P_f^{ADIS} (Error)	Ref.
1	$g = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6 + 0.001\sum_{i=1}^{6}\sin(100x_i)$	$x_{1...4}$: LN(120,12) x_5: LN(50,15) x_6: LN(40,12)	Linear LS with noise term	1.22e-02	1.19e-02 (-2.4 %)	[4]
2	$g = x_1x_2 - 146.14$	x_1: N(78064.4, 11709.7) x_2: N(0.0104, 0.00156)	Multiple design points	1.46e-07	1.61e-07 (10.8 %)	[4]
3	$g = 2 + 0.015\sum_{i=1}^{9}x_i^2 - x_{10}$	$x_{1...10}$: N(0, 1)	Quadratic LS 10 terms	5.34e-03	5.25e-03 (-1.7 %)	[3]
4	$g = 0.1(x_1 - x_2)^2 - \frac{(x_1 + x_2)}{\sqrt{2}} + 2.5$	x_1: N(0, 1) x_2: N(0, 1)	Quadratic LS with mixed term, convex LS	4.16e-03	3.69e-03 (-11.3 %)	[17]
5	$g = -0.5(x_1 - x_2)^2 - \frac{(x_1 + x_2)}{\sqrt{2}} + 3$	x_1: N(0, 1) x_2: N(0, 1)	Concave LS	1.05e-01	1.02e-01 (-2.1 %)	[17]
6	$g = 2 - x_2 - 0.1x_1^2 + 0.06x_1^3$	x_1: N(0, 1) x_2: N(0, 1)	Nonlinear LS with saddle point	3.47e-02	3.04e-02 (-12.4 %)	[3]
7	$g = 2.5 - 0.2357(x_1 - x_2) + 0.00463(x_1 + x_2 - 20)^4$	x_1: N(10, 3) x_2: N(10, 3)	Highly nonlinear LS	2.86e-03	3.10e-03 (8.4 %)	[18]
8	$g = 3 - x_2 + (4x_1)^4$	x_1: N(0, 1) x_2: N(0, 1)	Highly nonlinear LS	1.80e-04	1.91e-04 (6.0 %)	[19]
9	$g_1 = 2.677 - x_1 - x_2$ $g_2 = 2.500 - x_2 - x_3$ $g_3 = 2.323 - x_3 - x_4$ $g_4 = 2.250 - x_4 - x_5$ $g = \max(g_1, g_2, g_3, g_4)$	$x_{1...5}$: N(0, 1)	Parallel system	2.11e-04	2.02e-04 (-4.4 %)	[19]
10	$g_1 = -x_1 - x_2 - x_3 + 3\sqrt{3}$ $g_2 = -x_3 + 3$ $\min(g_1, g_2)$	x_1: N(0, 1) x_2: N(0, 1) x_3: N(0, 1)	Series system	2.57e-03	2.72e-03 (6.0 %)	[2]
11	$g_1 = -x_1 - x_2 - x_3 + 3\sqrt{3}$ $g_2 = -x_3 + 3$ $\max(g_1, g_2)$	x_1: N(0, 1) x_2: N(0, 1) x_3: N(0, 1)	Parallel system	1.23e-04	1.15e-04 (-6.3 %)	[2]
12	$g_1 = 2 - x_2 + \exp(-0.1x_1^2) + (0.2x_1)^4$ $g_2 = 4.5 - x_1x_2$ $\min(g_1, g_2)$	x_1: N(0, 1) x_2: N(0, 1)	Series system Multiple design points	3.54e-03	3.90e-03 (10.1 %)	[19]

Case	Limit-state function(s)	Stochastic variables	Description	P_f	P_f^{ADIS} (Error)	Ref.
13	$g_1 = 2 - x_2 + \exp\left(-0.1x_1^2\right) + (0.2x_1)^4$ $g_2 = 4.5 - x_1 x_2$ $\max(g_1, g_2)$	x_1: N(0, 1) x_2: N(0, 1)	Parallel system	2.50e-04	2.11e-04 (-15.6 %)	[19]
14	$g_1 = 0.1(x_1 - x_2)^2 - \frac{(x_1 + x_2)}{\sqrt{2}} + 3$ $g_2 = 0.1(x_1 - x_2)^2 + \frac{(x_1 + x_2)}{\sqrt{2}} + 3$ $g_3 = x_1 - x_2 + 3\sqrt{2}$ $g_4 = -x_1 + x_2 + 3\sqrt{2}$ $g = min(g_1, g_2, g_3, g_4)$	x_1: N(0, 1) x_2: N(0, 1)	Series system Multiple design points	4.492e-03	4.850e-03 (8.0 %)	[17]

The ADIS method is examined on

- *Efficiency*. This is reflected by the number of G-function evaluations necessary to obtain a converged solution. This number is compared with MCS and DS.
- *Robustness*. This reflects how the method performs in the case of a complex limit-state function: noisy failure function, highly nonlinear failure function, multiple design points and/or multiple failure functions.
- *Accuracy*. Does the method converge to the exact solution, provided that enough samples are taken into account.

Sampling is ended when the maximum relative error in the probability value is below a threshold value. Hence an equal accuracy level is obtained with all three sampling methods and therefore their efficiencies can be compared. The maximum relative error is given by:

$$E_{max}^{rel} = z_{\alpha/2} COV_{p_f} = \Phi^{-1}\left(\frac{\gamma + 1}{2}\right) COV_{P_f} \tag{10}$$

where γ is the confidence level and Φ the cumulative standard normal distribution. For each design point, the current value of the COV_{Pf} is checked against a threshold value, where COV_{Pf} is given by:

$$COV_{Pf} = \sqrt{\frac{1 - P_f}{N_{sim} P_f}} \tag{11}$$

For all problems the threshold coefficient of variation for the probability of failure COV_{Pf} was set to 0.1, which means that with 95 % confidence the relative error in the estimate of the probability of failure P_f is less than:

$$E_{max}^{p_f} = 1.96\,COV_{P_f} \approx 20\% \qquad (12)$$

This accuracy is acceptable for most engineering applications. In general, the real error will be less than 10 %, which is often better than the errors produced in other parts of the analysis (e.g. accuracy of the underlying deterministic and/or numerical model). Reducing the COV value reduces the error at the expense of more simulation.

Table 2 Number of deterministic analyses required by the different stochastic methods.

Case	MCS	DS	ADIS
1	7655	2680	136
2	$> 10^9$	107	57
3	17830	11216	116
4	27096	999	65
5	942	190	32
6	2734	307	81
7	36835	1502	71
8	354130	4039	148
9	361701	19688	91
10	37659	859	51
11	563723	2504	63
12	24902	212	32
13	351660	1573	65
14	41220	448	28

3.1 Discussion

The various problems serve to demonstrate the accuracy, efficiency and robustness of the ADIS method. In all problems the same default settings for $\Delta\beta$=0.1 and *Pratio*=0.4, presented in section 2, have been used. As previously stated, all results have a similar accuracy level by selecting a fixed value for the coefficients of variation of P_f.

The value obtained for the probability of failure with the ADIS method, given the above accuracy level, is presented in column 6 of Table 1 together with the relative error in P_f between parentheses. This error was (well) below the maximum expected error of 20% for all problems. The ADIS method therefore produced accurate results.

The required number of simulations is presented in Table 2, columns 2 to 4, for respectively, MCS, DS and ADIS. For all problems ADIS proves to be very efficient, requiring (much) less

than 200 simulations to determine the failure probability for specified accuracy level. Problem 3 reflects the efficiency in the case of a larger number of variables.

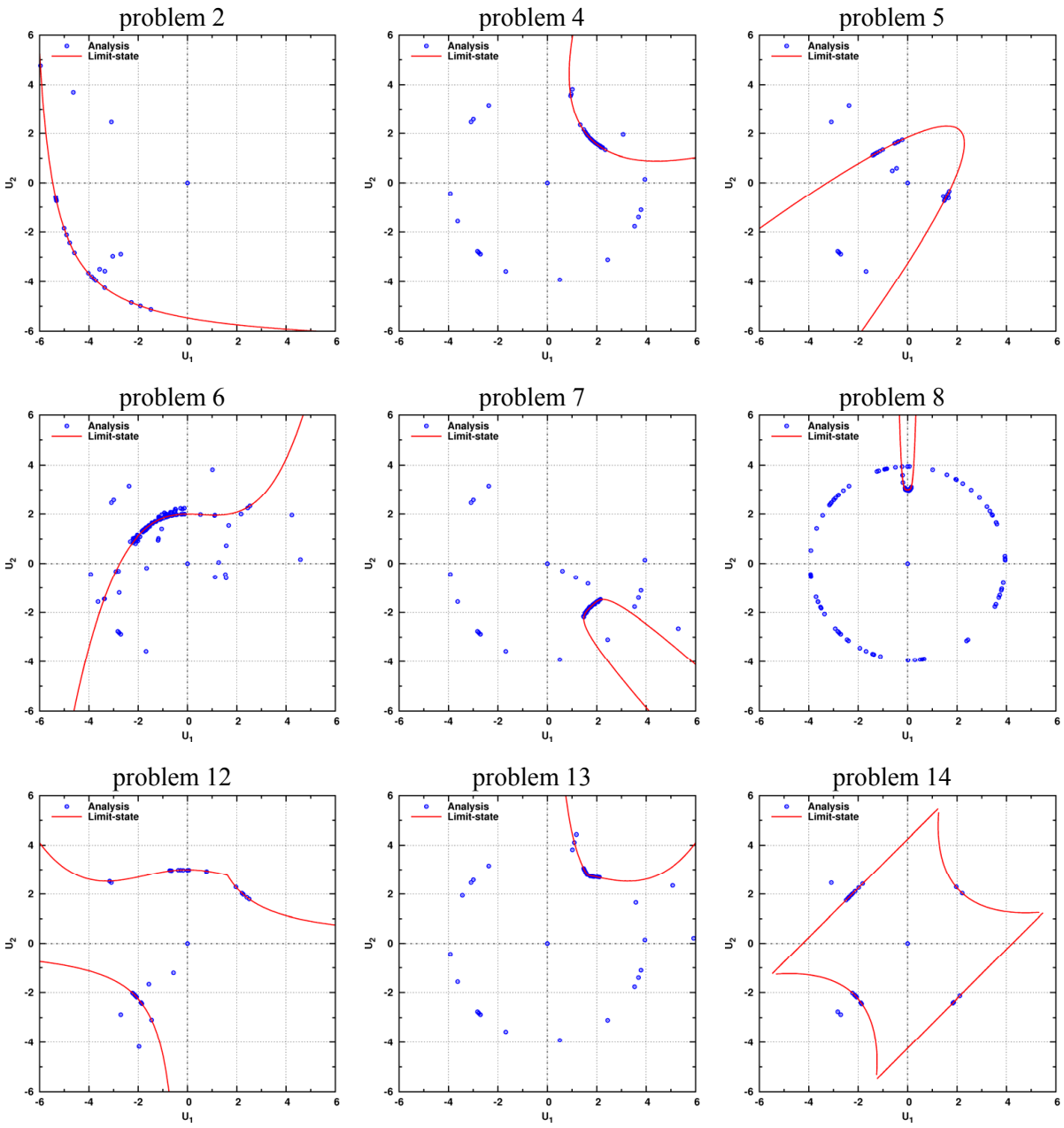

Figure 4 Sample plots for two-dimensional test problems.

Robustness is demonstrated by the noisy limit state of problem 1, multiple design points of problem 2, highly nonlinear limit states of problems 6 to 8; multiple failure functions of problems 9 to 14, in which multiple design points are present for problems 12 and 14 as well. Problem 14 has been slightly modified compared to the original problem, in order to obtain a

more complex situation with four equally significant design points. For all problems the ADIS method proved to be robust.
Figure 4 shows sample plots obtained with ADIS for the two-dimensional problems, clearly demonstrating the approach. The sample points that lie on a sphere, see for example problems 4, 7, 8 and 13, relate to the exact line-search at start of the DS method before a response surface could be fit. In addition, all problems show a clustering of points around the most important parts of the limit state.
Because of the above characteristics, ADIS is of particular interest in applications with a low probability of failure and medium number (up to about 40) of stochastic variables, such as in structural reliability analyses of aircraft structure.

3.2 Dimensionality test problem

To demonstrate the efficiency of the ADIS method for higher dimensional problems, the following test problem was evaluated:

$$g = 3\sqrt{n} - \sum_{i=1}^{n} x_i \qquad (13)$$

where n is the number of dimensions and x_i are standard normal variables N(0,1).
The limit state is a linear hyper-plane at a fixed distance $\beta = 3$ from the origin, yielding a fixed β-sphere radius and corresponding probability of failure $P_f = \Phi(-\beta) = 0.00135$. For various number of dimensions up to 50, the probability of failure has been computed with the ADIS method and compared with the solution obtained with FORM which provides an exact solution for this problem. The obtained efficiency, number of required G-function evaluations (deterministic analyses), is depicted in Figure 5 for both ADIS (filled-circle symbol) and FORM (open-circle symbol). In this case, both methods prove to be efficient even for high dimensions and ADIS is even somewhat more efficient than FORM. The number of G-function evaluations to build the initial response surface is depicted by the line with triangles resulting in a linear response surface which suffices for this problem. The number of additional G-function evaluations needed by ADIS to arrive at a converged solution (here $COV_\beta = 0.01$) is given by the difference between this line and the line with circles and shows a linear behaviour for increasing dimensions as well. These additional G-function evaluations represent the exact line-searches in the most important directions, to improve the accuracy of the failure probability and is used to update the response surface as well. The latter is not of importance in this particular example due to the linearity of the problem, but is important in general.
The required number of G-function evaluations to arrive at an incomplete (diamond symbol) and full quadratic (square symbol) response surface is depicted in the figure as well. For the

latter, the number of function evaluations increases considerably demonstrating the curse of dimensionality. As such, the ADIS method is of particular interest in applications with a low probability of failure and medium number (up to about 40) of stochastic variables. For many real applications (for instance in aircraft industry) the number of most important random variables (e.g. material properties and loading) will be (much) less than 40. Furthermore, the characterisation of this number of distribution functions already poses a difficult task.

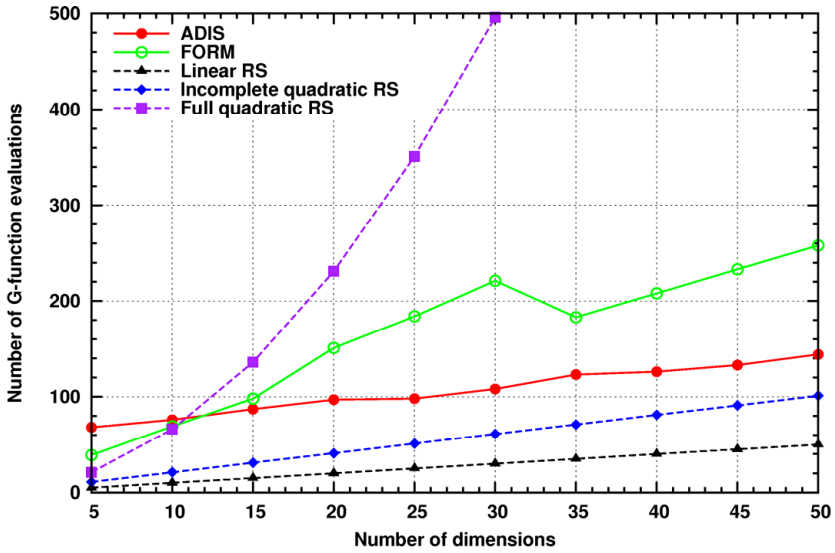

Figure 5 Number of samples (efficiency) versus the number of problem dimensions.

4 Conclusion

Importance sampling methods are more efficient than Monte Carlo Simulation and Directional Simulation, but require information about the location of the limit state(s), especially the part closest to the origin in U-space. Gathering this information can be expensive and can fail to locate all the important parts. In this paper, a new importance sampling method ADIS has been presented combining directional simulation and a response surface approach in an efficient adaptive scheme. The method has been demonstrated on a broad range of complex limit states that can occur in practice, of which several that pose potential problems to stochastic methods in general, and proved to be very efficient, accurate and robust. For this reason the method is of particular interest in applications with a low probability of failure and medium number (up to about 40) of stochastic variables, such as structural reliability in aircraft industry.

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