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ABSTRACT
This document contains the (mathematical part of the) contribution to the book "The Mathematics of Systems and Control: From Intelligent Control to Behavioral Sytems"(1999), dedicated to Jan C. Willems[1] on the occasion of his 60th birthday.

A brief overview is given of the behavioural approach to modelling linear inequality systems. Linear inequality systems often arise when models of aerospace systems are derived from so-called first principles.

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On behavioural inequalities

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Summary

This document contains the (mathematical part of the) contribution to the book 'The Mathematics of Systems and Control: From Intelligent Control to Behavioral Systems' (1999), dedicated to Jan C. Willems[1] on the occasion of his 60th birthday.

A brief overview is given of the behavioural approach to modelling linear inequality systems. Linear inequality systems often arise when models of aerospace systems are derived from so-called first principles.

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1 Introduction

Mathematical models of constrained dynamical systems often contain higher-order differential equations to model the dynamics. Additional inequalities may be present that model the hard environmental or operational restrictions. Given the fact that both authors worked that long in the vicinity of Jan Willems it is only natural that they studied constrained systems in a behavioural setting. Behavioural theory is well developed, especially for linear dynamical systems [9, 10, 11].

In the behavioural theory, which encompasses a foundation for the theory of deterministic dynamical systems, all variables are initially treated on an equal footing whereas in a more classical setting usually an explicit distinction is made between input and output variables. Such a distinction however, may not be clear a priori. Examples that illustrate this already in a linear context can be found in for instance [11]. An other example is a unilaterally constrained robotic manipulator [4]. Interaction of a manipulator with its environment, for instance grapple of an object, will inevitably mean restrictions on (possibly all) the positions, velocities and forces that are used to model the behaviour of the manipulator.

An important feature of the behavioural framework is that it offers a mathematical theory to discuss interconnected systems. In the behavioural theory a basic distinction is made between a system and its representation(s). In this paper the beginning of a theory is presented for systems described by difference inequalities in a behavioural setting. A dynamical system Σ is defined as a triple $\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T} \subseteq \mathbb{R}$ is the time set, $\mathbb{W}$ is the space of variables, and $\mathfrak{B}$ is a subset of $\mathbb{W}^{\mathbb{T}}$. $\mathfrak{B}$ called the behaviour of the system. Detailed introductions to the behavioural approach to systems and control can be found in [9, 10, 11].

The remainder of this paper is organized as follows. In section 2 we will introduce the notion of convex conical systems. In section 3 we formally define unilateral dynamical systems. It is investigated which properties allow dynamical systems to be described by a class of difference inequalities. In section 4 we discuss the elimination problem for difference inequalities. In section 5 results on a Farkas theorem for behavioural inequalities are presented. Concluding remarks can be found in section 6. References are collected in section 7.

2 Convex conical behaviours

We will discuss dynamical systems that can be described by linear difference inequalities. Some basic notions and properties are introduced, where we closely follow the line of thought presented in [7] for static inequalities.

Definition 2.1 *A dynamical system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ *is said to be convex if* $\mathbb{W}$ *is a real vector space over* $\mathbb{R}$ *and* $\mathfrak{B}$ *is a convex subset of* $\mathbb{W}^{\mathbb{T}}$*, i.e. if* $w_1, w_2 \in \mathfrak{B}$ *then* $\{(1-\alpha)w_1 + \alpha w_2 | 0 \leq \alpha \leq 1\} \in \mathfrak{B}$*. It is said to be conical if* $\mathfrak{B}$ *is a cone, i.e.* $\mathfrak{B}$ *is closed under multiplication by a nonnegative scalar:* $\{w \in \mathfrak{B}, \alpha \geq 0\} \Rightarrow \{\alpha w \in \mathfrak{B}\}$*. If a system is both convex and conical it is called convex conical.* □

Linear systems are a special case of convex conical systems. (Unless stated otherwise, proofs of results presented in this paper can be found in [3].)

Proposition 2.2 *Let* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ *be a dynamical system. Then* Σ *is convex conical if and only if* $\mathfrak{B}$ *contains all nonnegative linear combinations of its elements, i.e. if* $w_1, ..., w_n \in \mathfrak{B}$ *and* $\alpha_1, ..., \alpha_n \in [0, \infty)$ *then* $\alpha_1 w_1 + ... + \alpha_n w_n \in \mathfrak{B}$. □

For a behaviour $\mathfrak{B}$ let $(-\mathfrak{B}) := \{w \in \mathbb{W}^{\mathbb{T}} \mid -w \in \mathfrak{B}\}$.

Proposition 2.3 *Let* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ *be a convex conical dynamical system. Then:*

(i) The behaviour of the smallest linear system containing Σ*, denoted by* $\Sigma^{\mathcal{L}}$*, is given by* $\mathfrak{B} - \mathfrak{B} = \{w \in \mathbb{W}^{\mathbb{T}} \mid w = w_1 - w_2,\ w_1, w_2 \in \mathfrak{B}\}$.

(ii) The behaviour of the largest linear system contained in Σ*, denoted by* $\Sigma_{\mathcal{L}}$*, is given by* $\mathfrak{B} \cap (-\mathfrak{B}) = \{w \in \mathbb{W}^{\mathbb{T}} \mid w \in \mathfrak{B} \text{ and } w \in (-\mathfrak{B})\}$. □

In $\mathbb{R}^n$ a subset is called convex polyhedral if it is the intersection of a finite collection of closed halfspaces.

Definition 2.4 *Let* $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$ *be a discrete-time dynamical system. Then* Σ *is said to be a finite-polyhedral (conical) system if* $\forall t_1, t_2 \in \mathbb{Z}, -\infty < t_1 \leq t_2 < \infty$, $\mathfrak{B}|_{[t_1,t_2]}$ *is (a) polyhedral (cone) in* $(\mathbb{R}^q)^{t_2-t_1+1}$*. In that case* $\mathfrak{B}$ *is said to be (a) finite-polyhedral (cone).* □

Proposition 2.5 *Let* $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$ *be a time-invariant discrete-time dynamical system. Let* Σ *be finite-polyhedral. Then* Σ *is complete if and only if* $\mathfrak{B}$ *is closed.* □

The question arises if the 'finite-polyhedral' condition on Σ is a necessary condition in proposition 2.5. We return to this issue later.

3 Behavioural inequalities

A behavioural difference inequality representation of a discrete-time dynamical system with the time axis $\mathbb{T} = \mathbb{Z}$ and signal space $\mathbb{W}$ is, as in the linear case, defined by two integers L and l, and a map $f : \mathbb{W}^{L-l+1} \to \mathbb{R}^g$ (for some $g \in \mathbb{N}$). (In the remainder, equalities as well as inequalities will be referred to as equations.) A difference inequality is given by:

$$f(w(t+L), ..., w(t+l)) \geq 0, \forall t \in \mathbb{Z}. \tag{1}$$

It is clear that the system $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$, with

$$\mathfrak{B} = \{w : \mathbb{Z} \to \mathbb{W} \mid \text{ equation (1) is satisfied}\} \tag{2}$$

defines a time-invariant dynamical system.

Definition 3.1 *Let $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$ be a discrete-time dynamical system. If there exists a map $f : \mathbb{W}^{L-l+1} \to \mathbb{R}^g$ such that $\mathfrak{B}$ allows a representation as in (2) then the system Σ is said to be a unilateral dynamical system.* □

We will focus on convex conical unilateral dynamical systems whose behaviour can be represented by

$$\mathfrak{B} = \{w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid R(\sigma, \sigma^{-1})w \geq 0\}, \tag{3}$$

where σ denotes the shift operator and $R \in \mathbb{R}^{g \times q}[s, s^{-1}]$ the $(g \times q)$ polynomial matrices with real coefficients and positive and negative powers of the indeterminate s. As in the linear case [9], the number of columns of $R(s, s^{-1})$ is fixed and equals the number of manifest variables. The number of rows of $R(s, s^{-1})$ is equal to the number of equations used to describe the behaviour.

The question immediately arises what intrinsic properties of a dynamical system allow its behaviour to be represented as in (3). In [9] it has been shown that a discrete-time dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ can be described by a difference equality if and only if it is linear, time-invariant and complete.

It is clear that system $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$ with $\mathfrak{B}$ as in (3) is a convex conical system. Convex conical behaviours can also arise from certain nonlinear representations. For instance, the discrete-time system that is described by the nonlinear latent variable description $\{w(t) = \ell^2(t)\}$, can also be described by the convex conical manifest description $\{w(t) \geq 0\}$.

To differentiate behaviours where inequalities are present from behavioural representations where only equalities are present we introduce the following notation. Let $R_1 \in \mathbb{R}^{g\times q}[s, s^{-1}]$ and $R_2 \in \mathbb{R}^{r\times q}[s, s^{-1}]$. Denote:

$$\begin{aligned} \mathfrak{B}_E(R_1) &:= \{w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid R_1(\sigma, \sigma^{-1})w = 0\}, \\ \mathfrak{B}_I(R_2) &:= \{w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid R_2(\sigma, \sigma^{-1})w \geq 0\}. \end{aligned} \tag{4}$$

The following definition is a generalization of the notion of lineality space for the static case [7].

Definition 3.2 *The lineality behaviour $\mathfrak{B}_{\mathcal{L}}$ of a nonempty convex conical system $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$ is defined as $\mathfrak{B}_{\mathcal{L}} = \mathfrak{B} \cap (-\mathfrak{B})$. For system Σ, the system $\Sigma_{\mathcal{L}} = (\mathbb{Z}, \mathbb{W}, \mathfrak{B}_{\mathcal{L}})$ is called the lineality system.* □

By proposition 2.3 the lineality system is the largest linear system in a given system.

Lemma 3.3 *Let $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$ be a discrete-time dynamical system with $\mathfrak{B} = \mathfrak{B}_I(R)$. Then $\mathfrak{B}_{\mathcal{L}} = \mathfrak{B}_E(R)$.* □

Difference inequalities uniquely define the lineality behaviour. Since the reverse statement is not generally true one can not hope to find a difference inequality representation of a behaviour from a characterization of its lineality behaviour. This implies that the results that have been obtained in the behavioural approach to linear difference equalities are not directly applicable to the inequality case. To obtain an inequality description a different approach must be followed.

Definition 3.4 *Let α be a $\mathbb{R}^q$-valued sequence with compact support. Then the set $\mathcal{H} = \{w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid \sum_{t\in\mathbb{Z}} \alpha^T(t)w(t) \geq 0\}$ is said to be a halfspace in $(\mathbb{R}^q)^{\mathbb{Z}}$. If a set $\mathcal{P}$ is the intersection of a finite number of halfspaces $\mathcal{H}_i$, i.e. $\mathcal{P} = \cap_{i=1}^{g}\mathcal{H}_i$ for some $g \in \mathbb{N}$, then the set $\mathcal{P}$ is said to be a polyhedral cone in $(\mathbb{R}^q)^{\mathbb{Z}}$.* □

Note that if the sequence α in definition 3.4 has compact support in $[t_*, t^*]$, then for $w \in \mathcal{P}$ there is no requirement on w outside this interval $[t_*, t^*]$. For a polyhedral cone $\mathcal{P}$ in $(\mathbb{R}^q)^{\mathbb{Z}}$, define $\sigma^t\mathcal{P}, t \in \mathbb{Z}$, by $\sigma^t\mathcal{P} := \{\sigma^t w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid w \in \mathcal{P}\}$. This leads to the following notion.

Definition 3.5 *Let* $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ *be a discrete-time convex time-invariant complete dynamical system. Then* Σ *is said to be shifted-polyhedral conical if there exists a polyhedral cone* $\mathcal{P}$ *in* $(\mathbb{R}^q)^{\mathbb{Z}}$ *such that* $\mathfrak{B} = \cap_{t\in\mathbb{Z}}\sigma^t\mathcal{P}$. *In that case* $\mathfrak{B}$ *is said to be a shifted-polyhedral cone.* □

Theorem 3.6 *Let* $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$ *be a discrete-time convex finite-polyhedral time-invariant complete dynamical system. Then:* Σ *is shifted-polyhedral conical if and only if* $\exists R \in \mathbb{R}^{g\times q}[s, s^{-1}]$ *such that* $\mathfrak{B} = \{w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid R(\sigma, \sigma^{-1})w \geq 0\}$. □

It follows from definition 3.1 that if $\Sigma = (\mathbb{Z}, (\mathbb{R}^q)^{\mathbb{Z}}, \mathfrak{B})$ satisfies the conditions in theorem 3.6 with $\mathfrak{B}$ a shifted-polyhedral cone then Σ is a unilateral dynamical system. The question arises if the conditions $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ is complete, time-invariant and convex finite-polyhedral conical, or equivalently (by proposition 2.5), $\mathfrak{B}$ is closed, shift-invariant and a convex finite-polyhedral cone, are also sufficient for Σ to be a shifted polyhedral cone. This conjecture, which was raised in [1], was disproven in [6] for the case $\mathbb{T} = \mathbb{Z}_+$ by the following illustrative and nontrivial counterexample.

Example 3.7 *Define the following sequence of vectors in* $(\mathbb{R})^{\mathbb{Z}+}$:

$$\begin{aligned} k_1 &:= (2, 0, 0, \ldots), \\ k_2 &:= (1, 2, 0, 0, \ldots), \\ k_n &:= (\underbrace{1, 1, \ldots, 1}_{(n-1)\text{-times}}, 2, 0, 0, \ldots), \end{aligned} \tag{5}$$

i.e. $k_n(i) = 1$ *if* $1 \leq i < n$, $k_n(n) = 2$ *and* $k_n(i) = 0$ *if* $i > n$. *Now define the following sequence of polyhedral cones in* $(\mathbb{R})^{\mathbb{Z}+}$:

$$\begin{aligned} K_1 &:= \mathit{cone}(k_1) := \{w \in (\mathbb{R})^{\mathbb{Z}+} \mid w = \lambda_1 k_1, \lambda_1 \geq 0\}, \\ K_m &:= \mathit{cone}(k_1, k_2, \ldots, k_m) := \{w \in (\mathbb{R})^{\mathbb{Z}+} \mid w = \textstyle\sum_{i=1}^m \lambda_i k_i, \lambda_i \geq 0\}. \end{aligned} \tag{6}$$

Now define system $\Sigma = (\mathbb{Z}_+, \mathbb{W}, \mathfrak{B})$ *with*

$$\mathfrak{B} = \overline{\cup_{n\geq 1} K_n}, \tag{7}$$

i.e. the closure of the union of the cones K_i. *First we will show that* $\mathfrak{B}$ *is closed, shift-invariant* *and finite-polyhedral. Now by construction in (7)* $\mathfrak{B}$ *is closed. From* $\sigma K_m = K_{m-1}$ *we have* $\sigma\mathfrak{B} = \mathfrak{B}$. *It remains to show that* $\mathfrak{B}$ *is finite-polyhedral. It suffices to show that the projection of* $\mathfrak{B}$ *on the first* n *coordinates of* $\mathbb{Z}_+$ *is equal to* $cone((1,0,\ldots,0)^T,(1,2,0,\ldots,0)^T,\ldots,(1,1,\ldots,1,2)^T) \subseteq \mathbb{R}^n$. *The latter statement follows from the fact that in* $\mathbb{R}^n$*:*

$$(1,1,\ldots,1) = \frac{1}{2}(1,1,\ldots,1,2) + \frac{1}{4}(1,1,\ldots,1,2,0) + \ldots + \frac{1}{2^n}(2,0,\ldots,0). \tag{8}$$

To disprove the conjecture it must be shown that there is no polynomial matrix $R(s)$ *such that:* $w \in \mathfrak{B} \Leftrightarrow R(\sigma)w \geq 0$. *Let* $\tilde{K}_m$ *be the projection of* K_m *on the first* m *coordinates of* $\mathbb{Z}_+$*, i.e.*

$$\tilde{K}_m := cone((1,0,\ldots,0)^T,(1,2,0,\ldots,0)^T,\ldots,(1,1,\ldots,1,2)^T). \tag{9}$$

It is easy to see that from $(x_1,x_2,\ldots,x_{n+1})^T \in \tilde{K}_{n+1}$ *it follows that* $(x_1,x_2,\ldots,x_n)^T \in \tilde{K}_n$ *and* $(x_2,x_3\ldots,x_{n+1})^T \in \tilde{K}_n$. *However,* $(x_1,x_2,\ldots,x_n)^T,(x_2,x_3\ldots,x_{n+1})^T \in \tilde{K}_n$ *does not imply that* $(x_1,x_2,\ldots,x_{n+1})^T \in \tilde{K}_{n+1}$. *To see this, observe that* $(1,1,\ldots,1)^T \in \tilde{K}_n$, *and* $(1-\frac{1}{2^{n-1}},1,1,\ldots,1)^T \in \tilde{K}_n$. *Now consider, in* $\mathbb{R}^{n+1}$, *the vector* $(1-\frac{1}{2^{n-1}},1,1,\ldots,1)^T$. *Suppose that this vector is in* $\tilde{K}_{n+1}$. *Then from*

$$\begin{aligned}(1-\tfrac{1}{2^{n-1}},1,1,\ldots,1) &= \tfrac{1}{2}(1,1,\ldots,2) + \tfrac{1}{4}(1,1,\ldots,1,2,0) + \ldots \\ &\quad + \tfrac{1}{2^n}(1,2,0,\ldots,0) - \tfrac{1}{2^{n+1}}(2,0,\ldots,0),\end{aligned} \tag{10}$$

it follows that $\lambda_1 < 0$. *This contradicts the requirement that* $\lambda_1 \geq 0$.

Now assume that there does exist a polyhedral cone $K \in \mathbb{R}^n$*, for some* $n \in \mathbb{N}$*, such that*

$$\{w \in \mathfrak{B}\} \Leftrightarrow \{\forall t \in \mathbb{Z}_+,\ (w(t),w(t+1),\ldots,w(t+n-1))^T \in K\}. \tag{11}$$

By construction one has that $\tilde{K}_n \subseteq K$. *But now from* $K_n \subseteq K_{n+1}, \forall n \in \mathbb{N}$*, and the fact that* $(x_1,x_2,\ldots,x_n)^T \in \tilde{K}_n$ *and* $(x_2,x_3\ldots,x_{n+1})^T \in \tilde{K}_n$ *does not imply that* $(x_1,x_2,\ldots,x_{n+1})^T \in \tilde{K}_{n+1}$ *it follows immediately that one can not conclude from* $(w(t),w(t+1),\ldots,w(t+n-1))^T \in K$ *and* $(w(t+1),w(t+2),\ldots,w(t+n))^T \in K$ *that* $(w(t),w(t+1),\ldots,w(t+n))^T \in \mathfrak{B}|_{[t,t+n]}$. *This disproves the statement in equation (11). This in turn disproves the conjecture for*

$\mathbb{T} = \mathbb{Z}_+$. □

The above example can be used also to disprove the conjecture for the case $\mathbb{T} = \mathbb{Z}$, by redefining $k_n := (\ldots, 1, 1, 2, 0, 0, \ldots)$, where $k_n(n) = 2$.

In [9] the linear case is proven in two different ways. In the first proof, one of the essential observations is that a decreasing sequence of linear subpaces $\mathfrak{L}_t \in \mathbb{R}^q$ with $\mathfrak{L}_{t+1} \subseteq \mathfrak{L}_t$ attains a limit in a finite number of steps. This however need not be the case for convex polyhedral sets. Moreover, the convex cone that is obtained as the limit of this sequence need not to be polyhedral. Take the ice-cream cone $K := \{(x, y, z)^T \in \mathbb{R}^3 \,|\, x^2 + y^2 \le z^2, z \ge 0\}$. There are infinitely many polyhedral cones that contain the closed convex cone K. (This follows already in $\mathbb{R}^2$, where a circle can be obtained as the limit of a sequence of polygons.) However, K itself is not polyhedral. For the second proof in [9], $\mathbb{E} := (\mathbb{R}^q)^{\mathbb{Z}}$ is equipped with the topology of pointwise convergence. The dual of $\mathbb{E}$, denoted by $\mathbb{E}^*$, consists of all $\mathbb{R}^q$-valued sequences that have compact support, and is equipped with the weakest topology such that $(\mathbb{E}^*)^* = \mathbb{E}$. It is then shown that with $\mathfrak{B}^\perp = \{r \in \mathbb{E}^* \mid \sum_{t \in \mathbb{Z}} r(t) w(t) = 0\}$, $(\mathfrak{B}^\perp)^\perp = \mathfrak{B}$. In the present case we have so far not been able to prove or disprove that similar statements hold for inequality behaviours.

Open Problem 3.8 *Let $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$ be a discrete-time convex finite-polyhedral conical time-invariant dynamical system. Give necessary and sufficient conditions for $\mathfrak{B}$ to be a shifted-polyhedral cone.* □

We conclude that at least until problem 3.8 is solved, the finite-polyhedral condition on Σ can not be omitted easily from proposition 2.5.

4 The elimination problem

In this section we focus on elimination of latent variables from a difference inequality representation. Let $R_1 \in \mathbb{R}^{g\times q}[s,s^{-1}]$, $R_2 \in \mathbb{R}^{g\times d}[s,s^{-1}]$, $w : \mathbb{Z} \to \mathbb{R}^g$ and $\ell : \mathbb{Z} \to \mathbb{R}^d$. Consider the latent variable difference inequality:

$$R_1(\sigma,\sigma^{-1})w \geq R_2(\sigma,\sigma^{-1})\ell. \tag{12}$$

The question arises whether or not the latent variable ℓ can be eliminated from (12) to arrive at a representation $R(\sigma,\sigma^{-1})w \geq 0$ for some polynomial matrix R.

The next result states that the number of equations necessary to describe the manifest behaviour depends on the values of reals present in the latent variable description.

Proposition 4.1 *Let $a_i \in \mathbb{R}(i \in \underline{g})$. Consider the latent variable system $\Sigma_\ell = (\mathbb{Z}, (\mathbb{R}^q)^{\mathbb{Z}}, (\mathbb{R})^{\mathbb{Z}}, \mathfrak{B}_\ell)$, with $\mathfrak{B}_\ell = \{(w,\ell) \in (\mathbb{R}^q)^{\mathbb{Z}} \times (\mathbb{R})^{\mathbb{Z}} \mid R_1(\sigma,\sigma^{-1})w \geq a_1\ell, \ldots, R_g(\sigma,\sigma^{-1})w \geq a_g\ell\}$, $g \in \mathbb{N}$. Define $H_+ := \{i : a_i > 0\}$, $H_0 := \{i : a_i = 0\}$, $H_- := \{i : a_i < 0\}$, and $n_+ := card(H^+)$, $n_0 := card(H_0)$ and $n_- := card(H_-)$ (where card denotes cardinality.) The latent variables can be eliminated. Moreover, the manifest behaviour that the system Σ_ℓ represents can be described by the $(n_+ \cdot n_- + n_0)$ inequalities*

$$\begin{aligned} R_i(\sigma,\sigma^{-1})w &\geq 0,\ i \in H_0, \\ a_j R_k(\sigma,\sigma^{-1})w &\geq a_k R_j(\sigma,\sigma^{-1})w,\ j \in H_+, k \in H_-. \end{aligned}$$

Consequently, if $n_0 = 0$ and either $n_+ = 0$ or $n_- = 0$ then there are no restrictions on the manifest variables w. □

It is now easy to see that in case $R(\sigma,\sigma^{-1})w \geq A\ell$, with $A \in \mathbb{R}^{g\times d}$ and $\ell \in (\mathbb{R}^d)^{\mathbb{Z}}$, the latent variables can also be eliminated by repeated use of the result in proposition 4.1: write $\ell = col(\ell_1, \ldots, \ell_d)$ and eliminate the ℓ_i's one after the other. The resulting set of inequalities contains a large number of equations, but is stated in terms of the manifest variable w only. This also provides some clues for the elimination of the latent variables from $R(\sigma,\sigma^{-1})w \geq A(\sigma,\sigma^{-1})\ell$. For instance if $A(s,s^{-1})$ has positive or negative entries only, then the variables w are not restricted.

The following representation will play an important role in our discussion of the elimination problem. Let $b \in \mathbb{N}$, $M \in \mathbb{R}^{q\times b}[s, s^{-1}]$, and $N \in \mathbb{R}^{g\times b}[s, s^{-1}]$.

$$\begin{aligned} w &= M(\sigma, \sigma^{-1})\beta, \\ N(\sigma, \sigma^{-1})\beta &\geq 0. \end{aligned} \tag{13}$$

The variable β appears in (13) as a unilaterally constrained latent variable.

Proposition 4.2 *Let $\Sigma_\ell = (\mathbb{Z}, (\mathbb{R}^q)^{\mathbb{Z}}, (\mathbb{R}^d)^{\mathbb{Z}}, \mathfrak{B}_\ell)$ be a discrete-time time-invariant latent variable dynamical system represented by $R_1(\sigma, \sigma^{-1})w \geq R_2(\sigma, \sigma^{-1})\ell$, with $R_1 \in \mathbb{R}^{g\times q}[s, s^{-1}]$ and $R_2 \in \mathbb{R}^{g\times d}[s, s^{-1}]$. Then $\exists b \in \mathbb{N}$ and there are polynomial matrices $M \in \mathbb{R}^{q\times b}[s, s^{-1}]$ and $N \in \mathbb{R}^{g\times b}[s, s^{-1}]$ such that the manifest system $\Sigma = (\mathbb{Z}, (\mathbb{R}^q)^{\mathbb{Z}}, \mathfrak{B})$ of Σ_ℓ can be described by $\mathfrak{B} = \{w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid \exists \beta \in (\mathbb{R}^b)^{\mathbb{Z}}$ such that equation (13) holds$\}$.* □

Since a behaviour $\mathfrak{B}_I(R)$ is a special case of a latent variable description (take $R_2 = 0$ in (12)), proposition 4.2 applies to systems $\Sigma = (\mathbb{Z}, (\mathbb{R}^q)^{\mathbb{Z}}, \mathfrak{B}_I(R))$ as well. Note also that any representation (13) can be written as a latent variable description (12). Therefore, the remaining problem to be solved reads as follows. Can the latent variable β be eliminated from (13) to arrive at an inequality representation $R(\sigma, \sigma^{-1})w \geq 0$ for some polynomial matrix R? The following proposition gives sufficient conditions.

Proposition 4.3 *Let $M \in \mathbb{R}^{q\times b}[s, s^{-1}]$ and $N \in \mathbb{R}^{g\times b}[s, s^{-1}]$. Let $\Sigma_\beta = (\mathbb{Z}, (\mathbb{R}^q)^{\mathbb{Z}}, (\mathbb{R}^b)^{\mathbb{Z}}, \mathfrak{B}_\beta)$ be a discrete-time time-invariant latent variable dynamical system. Suppose it induces the manifest behaviour with $\mathfrak{B} = \{w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid \exists \beta \in (\mathbb{R}^b)^{\mathbb{Z}}$ such that (13) holds$\}$. Then there exists a polynomial matrix $R(s, s^{-1})$ such that the manifest behaviour of Σ_β is given by $\mathfrak{B} = \mathfrak{B}_I(R)$ if β is observable in $w = M(\sigma, \sigma^{-1})\beta$, or $N(s, s^{-1}) = 0$.* □

The first condition in proposition 4.3 implies that there exists a polynomial matrix $R'(s, s^{-1})$ such that $\beta = R'(\sigma, \sigma^{-1})w$. This gives $\{(I - MR')(\sigma, \sigma^{-1})w = 0, (NR')(\sigma, \sigma^{-1})w \geq 0\}$ as a model of the manifest behaviour. The second condition in proposition 4.3 implies that the prescribing equation is $w = M(\sigma, \sigma^{-1})\beta$. Since we are now in the linear case, the elimination theorem [9] provides us with the manifest behaviour $R''(\sigma, \sigma^{-1})w = 0$ for some polynomial matrix $R''(s, s^{-1})$.

Another promising approach, which uses results from the linear case, is given in [3]. Up till now we have not been able to formulate necessary conditions in terms of the matrices M and N.

5 On a Farkas theorem for behavioural inequalities

In this section we focus on efficient representations of behaviours that can be represented by

$$\mathfrak{B}_I(R) = \{w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid R(\sigma, \sigma^{-1})w \geq 0\}, \tag{14}$$

with $R \in \mathbb{R}^{g \times q}[s, s^{-1}]$. As in the linear case, minimality will always refer to keeping the number of equations as small as possible.

Definition 5.1 *Let $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$. Let $R \in \mathbb{R}^{g \times q}[s, s^{-1}]$ and $R' \in \mathbb{R}^{g' \times q}[s, s^{-1}]$. The systems of difference inequalities $R(\sigma, \sigma^{-1})w \geq 0$ and $R'(\sigma, \sigma^{-1})w \geq 0$ are said to be equivalent if $\{\mathfrak{B}_I(R) = \mathfrak{B}_I(R')\}$. The system of difference inequalities $R(\sigma, \sigma^{-1})w \geq 0$ is said to be a minimal inequality (or, for short, minimal) if: $\{\mathfrak{B}_I(R) = \mathfrak{B}_I(R')\} \Rightarrow \{g \leq g'\}$.* □

In for instance [9], it is shown that a kernel representation $R(\sigma, \sigma^{-1})w = 0$ is minimal if and only if $R(s, s^{-1})$ has full row-rank. It is easy to see that an inequality system $\{R(\sigma, \sigma^{-1})w \geq 0\}$ with $R \in \mathbb{R}^{1 \times q}[s, s^{-1}]$ is minimal, and also that $\mathfrak{B}_I(R) \neq \mathfrak{B}_E(R)$ if $R(s, s^{-1}) \neq 0$ and $R(s, s^{-1})$ has full row-rank.

Proposition 5.2 *The following holds:*

(i) Not every inequality behaviour $\mathfrak{B}_I(R)$ has a full row-rank inequality representation.

(ii) For every $q \in \mathbb{N}$, there exists an inequality system in q variables such that the minimum number of rows in the minimal inequality representation exceeds q. □

We are interested in the relation between two polynomial matrices $R(s, s^{-1})$ and $R'(s, s^{-1})$ when they satisfy

$$R(\sigma, \sigma^{-1})w \geq 0 \Rightarrow R'(\sigma, \sigma^{-1})w \geq 0. \tag{15}$$

Based on the static case, one may expect that such a relation should be the extension of Farkas' theorem to the behavioural case.

Proposition 5.3 *Let $R \in \mathbb{R}^{g \times q}[s, s^{-1}]$ be a full row-rank polynomial matrix. Let $R' \in \mathbb{R}^{g' \times q}[s, s^{-1}]$. Then: $\{R(\sigma, \sigma^{-1})w \geq 0 \Rightarrow R'(\sigma, \sigma^{-1})w \geq 0\}$ if and only if there exists a unique polynomial matrix $H \in \mathbb{R}_+^{g' \times g}[s, s^{-1}]$ such that $R'(s, s^{-1}) = H(s, s^{-1})R(s, s^{-1})$.* □

In order to extend proposition 5.3 to the general case, i.e. without the assumption that the polynomial matrix $R(s,s^{-1})$ has full row-rank, there remain some difficulties. One could try to extend the original proof given by Farkas in [5]. However, this proof explicitly uses the fact that every scalar that is unequal to zero is invertible. Such a general statement does not hold for elements of $\mathbb{R}^{g\times q}[s,s^{-1}]$. The most promising approach for the dynamic case seems to be the usage of mathematical tools such as the separation theorem of Hahn-Banach (see for instance [8]). The basic mathematical preliminaries read as follows. Denote $\mathbb{E} := (\mathbb{R}^q)^{\mathbb{Z}}$ with the topology of pointwise convergence. The dual of $\mathbb{E}$, denoted by $\mathbb{E}^*$, consists of all $\mathbb{R}^q$-valued sequences that have compact support. Let $R \in \mathbb{R}^{g\times q}[s,s^{-1}]$. Let $\mathfrak{B} = \mathfrak{B}_I(R)$. The polar cone of $\mathfrak{B}$, denoted by $\mathfrak{B}^{\#}$, is given by $\{w^* \in \mathbb{E}^* \mid \forall w \in \mathfrak{B} : \sum_{t\in\mathbb{Z}} w^*(t)w(t) \geq 0\}$. We would like to establish that $\mathfrak{B}^{\#} = \{w^* \in \mathbb{E}^* \mid \exists \alpha \in \mathbb{E}^*, \alpha \geq 0 \text{ such that } w^* = R^T(\sigma^{-1},\sigma)\alpha\}$, but we have so far not been able to prove or disprove these statements. The statements, together with the fact that $\{\mathfrak{B}_1 \subseteq \mathfrak{B}_2\}$ implies $\{\mathfrak{B}_2^{\#} \subseteq \mathfrak{B}_1^{\#}\}$ are believed to be useful in a proof of the following conjecture.

Conjecture 5.4 *Let* $R \in \mathbb{R}^{g\times q}[s,s^{-1}]$ *and* $R' \in \mathbb{R}^{g'\times q}[s,s^{-1}]$. *Then:* $\{R(\sigma,\sigma^{-1})w \geq 0 \Rightarrow R'(\sigma,\sigma^{-1})w \geq 0\}$ *if and only if there exists a polynomial matrix* $H \in \mathbb{R}_+^{g'\times g}[s,s^{-1}]$ *such that* $R'(s,s^{-1}) = H(s,s^{-1})R(s,s^{-1})$. □

When true, conjecture 5.4 would result in a Farkas theorem for behavioural inequalities.

The next logical step is to allow both equalities and inequalities in representation. For results in that direction we refer to [3].

6 Concluding remark

The beginning of a theory on dynamical systems described by behavioural difference inequalities has been presented. The notions of convexity and conicity have been introduced in a behavioural setting. It has been shown that so called shifted-polyhedral conical systems can be described by a difference inequality. We have also discussed the elimination problem for representations that involve latent variables. Sufficient conditions have been derived under which latent variables can be eliminated from a representation. The extension of Farkas' theorem has been proven for full row-rank polynomial matrices.

Throughout this paper we have shown that inequality systems have characteristics that are very different from characteristics of linear systems. Clearly, there remains a lot of work to be done on modelling behavioural inequalities, and a number of open research problems have been identified.

7 References

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