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Probabilistic Data Association Avoiding Track Coalescence

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Summary

For the problem of tracking multiple targets the Joint Probabilistic Data Association (JPDA) approach has shown to be very effective in handling clutter and missed detections. The JPDA, however, tends to coalesce neighbouring tracks and ignores the coupling between those tracks. Fitzgerald has shown that hypothesis pruning may be an effective way to prevent track coalescence. Unfortunately, this process leads to an undesired sensitivity to clutter and missed detections, and it does not support any coupling. To improve this situation, the paper follows a novel approach to combine the advantages of JPDA, coupling and hypothesis pruning into new algorithms. First, the problem of multiple target tracking is embedded into one filtering for a linear descriptor system with stochastic coefficients. Next, for this descriptor system the exact Bayesian and new JPDA filters are derived. Finally, through Monte Carlo simulations, it is shown that these new PDA filters are able to handle coupling and are insensitive to track coalescence, to clutter and to missed detections.

Abbreviations

ARTAS	Advanced Surveillance Tracker and Server
CPDA	Coupled Probabilistic Data Association
CPDA*	Track-coalescence-avoiding Coupled Probabilistic Data Association
EKF	Extended Kalman Filter
ENNPDA	Exact Nearest Neighbour Probabilistic Data Association
IMM	Interacting Multiple Model
JPDA	Joint Probabilistic Data Association
JPDA*	Track-coalescence-avoiding Joint Probabilistic Data Association
JPDAC	Joint Probabilistic Data Association - Coupled
MHT	Multiple Hypothesis Tracking
MLE	Maximum Likelihood Estimation
MMSE	Minimum Mean Square Error
MR	Mixture Reduction
MTA	Measurement to Track Association
MTMR	Multi Target Mixture Reduction
NLR	Nationaal Lucht- en Ruimtevaartlaboratorium
NN	Nearest Neighbour
PDA	Probabilistic Data Association
SME	Symmetric Measurement Equation

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Henk A.P. Blom and Edwin A. Bloem

Abstract— **For the problem of tracking multiple targets the Joint Probabilistic Data Association (JPDA) approach has shown to be very effective in handling clutter and missed detections. The JPDA, however, tends to coalesce neighbouring tracks and ignores the coupling between those tracks. Fitzgerald has shown that hypothesis pruning may be an effective way to prevent track coalescence. Unfortunately, this process leads to an undesired sensitivity to clutter and missed detections, and it does not support any coupling. To improve this situation, the paper follows a novel approach to combine the advantages of JPDA, coupling and hypothesis pruning into new algorithms. First, the problem of multiple target tracking is embedded into one filtering for a linear descriptor system with stochastic coefficients. Next, for this descriptor system the exact Bayesian and new JPDA filters are derived. Finally, through Monte Carlo simulations, it is shown that these new PDA filters are able to handle coupling and are insensitive to track coalescence, to clutter and to missed detections.**

Keywords— **Bayesian Filtering, Descriptor system, Joint Probabilistic Data Association, Multi-target tracking**

I. Introduction

FOR the problem of tracking multiple targets, the Joint Probabilistic Data Association (JPDA) filter [1] has shown to be very effective in handling clutter and missed detections. The JPDA, however, tends to coalesce neighbouring tracks. The aim of this paper is to develop probabilistic filters that both avoid JPDA's sensitivity to track coalescence and preserve JPDA's resistance to clutter and missed detections. This development forms a further elaboration of the new approach and filters presented in [2] and [3].

The Joint Probabilistic Data Association (JPDA) filter [1], [4] is the best known example of the Bayesian data association paradigm. From this point of view, JPDA seems to have a fundamental advantage over classical Measurement to Track Association (MTA) approaches, such as a single scan based Nearest Neighbour (NN) approach, or the multi-scan based Multiple Hypothesis Tracking (MHT) approach. Because of its appealing paradigm, JPDA has stimulated further developments, many of which have been directed to improving the stability or complexity of the numerical evaluation of the JPDA equations. This research has led to the development of several sub-optimal JPDA weight evaluation schemes [5], [6], [7], [8], [9] and to the Exact Nearest Neighbour version of the JPDA (ENNPDA) of [6]. The latter uses the JPDA weight evaluation and subsequently prunes all Gaussians from the conditional density, except the joint association hypothesis that has the heighest weight. The resulting ENNPDA appeared to be remarkably insensitive to track coalescence in case of no clutter and no missed detections. The dramatic pruning used for ENNPDA, however, leads to an undesired sensitivity to clutter and missed detections [10]. From this point of view, ENNPDA and JPDA seem to have complementary qualities, both of which one would like to combine into a new algorithm.

More fundamental studies have been directed toward the development of new approaches in approximating the conditional density. One direction is the approximation of the conditional density for each target's state by a reduced mixture of Gaussian densities, rather than by JPDA's single Gaussian. Through the introduction of appropriate distance measures the single-target Mixture Reduction (MR) scheme of [11] has been extended to a Multi Target Mixture Reduction (MTMR) version [12]. Another fundamental extension over JPDA is the estimation of the joint targets state. The underlying motivation is that the probabilistic sharing of measurements by closely spaced targets results in a correlation between the individual tracks [13]. Presently, this idea has been elaborated along three different directions, as follows.

- JPDA Coupled (JPDAC) filter approach. Following the JPDA derivation framework, a coupled filter has been given for the joint state (e.g. [14], pp. 328-329). During each filter cycle a single Gaussian replaces the Gaussian sum. The effectiveness of the JPDAC approach in combination with two other Bayesian approaches (Imaging sensor filter and IMM, respectively) has been demonstrated for closely spaced target situations [15], [16]. Note, however, that a direct comparison between JPDAC and JPDA has not been made in these papers.
- Symmetric Measurement Equation (SME) approach. Similar as JPDAC, SME yields a coupled filter for the joint state results. The novel idea is to transform the joint state observation equation such that the data association uncertainty disappears completely. The result of such transformation is a nonlinear filtering problem, which can be approached by e.g. an Extended Kalman Filter (EKF) for the joint target state. Effective SME transformations have been developed and some initial comparisons with JPDA have shown the effectiveness of the approach [17], [18], [19], [20], [21], [22]. Due to the nonlinear filtering for the joint targets' state, the SME approach evaluates the correlation between close target tracks. SME is numerically less complex than JPDA.
- Maximum Likelihood Estimation (MLE) approach. Here the aim is to recursively evaluate the MLE of the targets' joint state. By making use of a (single) Gaussian approximation around the maximum likelihood estimate of the joint targets' state, recursive methods for MLE based tracking have been developed [23], [24]. The most general

recursive MLE filter is based on a mean field approach towards the evaluation of the maximum likelihood estimate of the targets joint state [24]. The effectiveness of this recursive MLE approach relative to JPDA has been shown in [25], and two types of improvements over JPDA have been reported: (1) improved resistance to track coalescence, and (2) improved resistance to clutter measurements.

The JPDAC, the SME and the MLE developments all point into the direction that a study towards combining the complementary qualities of JPDA and ENNPDA into a new algorithm deserves a broader setup, in order to also try to combine the coupling between target tracks. To do so, this paper explores two complementary directions, the first of which provides a theoretical basis for incorporating coupling with JPDA, and the second exploits the ENNPDA advantages. The first direction starts by embedding the problem of multi-target tracking, given unassociated measurements from clutter and randomly detected targets, into one of filtering for a well-defined system of stochastic difference equations. The resulting embedding is a linear descriptor system with stochastic i.i.d. coefficients [2]. This representation forms the key to the development of the exact Bayesian filter equations for the conditional density of the joint state of the multiple targets, and of the Gaussian approximation-based filter algorithm. The latter filter algorithm appears to differ significantly from the JPDAC filter in case of missed detections, and is referred to as Coupled Probabilistic Data Association (CPDA) filter. The second, complementary direction shows how the hypothesis pruning approach of ENNPDA should be incorporated within the CPDA and JPDA filters. To do so, we first develop an ENNPDA-inspired track-coalescence-avoiding hypothesis pruning strategy. Applying this to CPDA and JPDA results into new algorithms called CPDA* and JPDA*, respectively, in which the * is short for "Track-coalescence-avoiding". In order to compare the newly developed CPDA, CPDA* and JPDA* filters versus each other and versus JPDAC, JPDA and ENNPDA, we subsequently run Monte Carlo simulations for a basic example in which track coalescence plays a distinctive role in JPDA. On the basis of these results, it appeared possible to characterize the differences between the various filters, to show the distinct advantage of the CPDA* and JPDA* algorithms over the other ones, and to identify the cause of track coalescence.

The paper is organized as follows. In section II, we introduce the multitarget tracking problem considered. In section III, we embed this problem into one of filtering given measurements from a linear descriptor system with stochastic coefficients. In section IV, we develop the CPDA filter algorithm and explain its difference with JPDAC and JPDA. In section V, we develop the track-coalescence-avoiding filter algorithms CPDA* and JPDA*. In section VI, we evaluate all new developments through Monte-Carlo simulations. Finally, in section VII, we summarize the results and draw more general conclusions.

II. The Stochastic Model

In this section we will describe the target model and the measurement model.

A. The target model

We consider M targets and we assume that the state of the ith target is modeled as follows :

$$x^i_{t+1} = a^i x^i_t + b^i w^i_t, \qquad i = 1, ..., M, \tag{1}$$

where x^i_t is the n-vectorial state of the i-th target, a^i and b^i are $(n \times n)$-matrices and w^i_t is a sequence of i.i.d. standard Gaussian variables of dimension n with w^i_t , w^j_t independent for all $i \neq j$ and w^i_t ,x^i_0, x^j_0 independent for all $i \neq j$. Let $x_t \triangleq \text{Col}\{x^1_t, ..., x^M_t\}$, $A \triangleq \text{Diag}\{a^1, ..., a^M\}$, $B \triangleq \text{Diag}\{b^1, ..., b^M\}$, and $w_t \triangleq \text{Col}\{w^1_t, ..., w^M_t\}$. Then we can model the state of our M targets as follows:

$$x_{t+1} = Ax_t + Bw_t \tag{2}$$

B. The measurement model

A set of measurements consists of two types of measurements, namely measurements originating from targets and measurements originating from clutter. First we will treat the two types of measurements separately. Subsequently we treat the random insertion of clutter measurements between the target measurements.

B.1 Measurements originating from targets

We assume that a potential measurement associated with state x^i_t (which we will denote by z^i_t) is modeled as follows:

$$z^i_t = h^i x^i_t + g^i v^i_t \qquad , i = 1, ..., M \tag{3}$$

where z^i_t is an m-vector, h^i is an $(m \times n)$-matrix and g^i is an $(m \times m)$-matrix, and v^i_t is a sequence of i.i.d. standard Gaussian variables of dimension m with v^i_t and v^j_t independent for all $i \neq j$. Moreover v^i_t is independent of x^j_0 and w^j_t for all i,j .

With $z_t \triangleq \text{Col}\{z^1_t, ..., z^M_t\}$, $H \triangleq \text{Diag}\{h^1, ..., h^M\}$, $G \triangleq \text{Diag}\{g^1, ..., g^M\}$, and $v_t \triangleq \text{Col}\{v^1_t, ..., v^M_t\}$, we obtain:

$$z_t = Hx_t + Gv_t \tag{4}$$

We next introduce a model that takes into account that not all targets have to be detected at moment t, which implies that not all potential measurements z^i_t have to be available as true measurements at moment t. To this end, we define the following variables: Let P^i_d be the detection probability of target i and let $\phi_{i,t} \in \{0,1\}$ be the detection indicator for target i, which assumes the value one with probability $P^i_d > 0$, independently of $\phi_{j,t}$, $j \neq i$. This result yields the following detection indicator vector ϕ_t:

$$\phi_t \triangleq \text{Col}\{\phi_{1,t}, ..., \phi_{M,t}\}.$$

Thus, the number of detected targets is $D_t \triangleq \sum_{i=1}^M \phi_{i,t}$. Furthermore, we assume that $\{\phi_t\}$ is a sequence of i.i.d.

vectors. In order to link the detection indicator vector with the measurement model, we introduce the following operator Φ: for an arbitrary (0,1)-valued M'-vector ϕ' we define $D(\phi') \triangleq \sum_{i=1}^{M'} \phi'_i$ and the operator Φ producing $\Phi(\phi')$ as a $(0,1)$-valued matrix of size $D(\phi') \times M'$ of which the ith row equals the ith non-zero row of Diag$\{\phi'\}$. Hence by defining, for $D_t > 0$,

$$\tilde{z}_t \triangleq \underline{\Phi}(\phi_t) z_t, \text{ where } \underline{\Phi}(\phi_t) \triangleq \Phi(\phi_t) \otimes I_m,$$

with I_m a unit-matrix of size m, and $\otimes$ denoting the tensor product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes I_m = \begin{bmatrix} aI_m & \vdots & bI_m \\ & \cdots\cdots & \\ cI_m & \vdots & dI_m \end{bmatrix}$$

we get the vector that contains all measurements originating from targets at moment t in a fixed order. In reality, however, we do not know the order of the targets. Hence, we introduce the stochastic $D_t \times D_t$ permutation matrix χ_t, which is conditionally independent of $\{\phi_t\}$. We also assume that $\{\chi_t\}$ is a sequence of independent matrices. Hence, for $D_t > 0$,

$$\tilde{\tilde{z}}_t \triangleq \underline{\chi}_t \tilde{z}_t, \text{ where } \underline{\chi}_t \triangleq \chi_t \otimes I_m,$$

is a vector that contains all measurements originating from targets at moment t in a random order.

B.2 Measurements originating from clutter

Let the random variable F_t be the number of false measurements at moment t. We assume that F_t has Poisson distribution:

$$\begin{aligned} p_{F_t}(F) &= \exp\left(-\lambda V\right) \tfrac{(\lambda V)^F}{F!}, && F = 0, 1, 2, \ldots \\ &= 0, && \text{else} \end{aligned}$$

where λ is the spatial density of false measurements (i.e. the average number per unit volume) and V is the volume of the validation region. Thus, λV is the expected number of false measurements in the validation gate. We assume that the false measurements are uniformly distributed in the validation region, which means that a column-vector v^*_t of F_t i.i.d. false measurements is assumed to have the following density:

$$p_{v^*_t|F_t}(v^*|F) = V^{-F}$$

where V is the volume of the validation region. Furthermore we assume that the process $\{v^*_t\}$ is a sequence of independent vectors, which are independent of $\{x_t\}, \{w_t\}, \{v_t\}$ and $\{\phi_t\}$.

B.3 Random insertion of clutter measurements

Let the random variable L_t be the total number of measurements at moment t. Thus,

$$L_t = D_t + F_t$$

With $\tilde{y}_t \triangleq \text{Col}\{\tilde{\tilde{z}}_t, v^*_t\}$, it follows with the above defined variables that

$$\tilde{y}_t = \begin{bmatrix} \underline{\chi}_t \underline{\Phi}(\phi_t) z_t \\ \cdots\cdots\cdots \\ v^*_t \end{bmatrix}, \text{ if } L_t > D_t > 0 \tag{5}$$

whereas the upper and lower subvector parts disappear for $D_t = 0$ and $L_t = D_t$ respectively. With this equation, the measurements originating from clutter still have to be randomly inserted between the measurements originating from the detected targets. To do so, we first introduce the following target indicator and clutter indicator processes, denoted by $\{\psi_t\}$ and $\{\psi^*_t\}$, respectively: Let the random variable $\psi_{i,t} \in$\{0,1\} be a target indicator at moment t for measurement i, which assumes the value one if measurement i belongs to a detected target and zero if measurement i comes from clutter. This result yields the following target indicator vector ψ_t of size L_t:

$$\psi_t \triangleq \text{Col}\{\psi_{1,t}, ..., \psi_{L_t,t}\}.$$

Let the random variable $\psi^*_{i,t} \in$\{0,1\} be a clutter indicator at moment t for measurement i, which assumes the value one if measurement i comes from clutter and zero if measurement i belongs to an aircraft (thus $\psi^*_{i,t} = 1 - \psi_{i,t}$). This result yields the following clutter indicator vector ψ^*_t of size L_t:

$$\psi^*_t \triangleq \text{Col}\{\psi^*_{1,t}, ..., \psi^*_{L_t,t}\}.$$

In order to link the target and clutter indicator vectors with the measurement model, we make use of the operator Φ introduced before. With this the measurement vector with clutter inserted reads as follows:

$$y_t = \left[\underline{\Phi}(\psi_t)^T \vdots \underline{\Phi}(\psi^*_t)^T\right] \tilde{y}_t \quad \text{if } L_t > D_t > 0 \tag{6}$$

Substituting (5) into (6) yields the following model for the observation vector y_t at moment t:

$$y_t = \left[\underline{\Phi}(\psi_t)^T \vdots \underline{\Phi}(\psi^*_t)^T\right] \begin{bmatrix} \underline{\chi}_t \underline{\Phi}(\phi_t) z_t \\ \cdots\cdots\cdots \\ v^*_t \end{bmatrix} \quad \text{if } L_t > D_t > 0 \tag{7}$$

This, together with equation (2), forms a complete characterization of our tracking problem in terms of stochastic difference equations.

III. Embedding into a descriptor system with stochastic coefficients

Because $\left[\underline{\Phi}(\psi_t)^T \vdots \underline{\Phi}(\psi^*_t)^T\right]$ is a permutation matrix for $L_t > D_t > 0$, its inverse satisfies

$$\left[\underline{\Phi}(\psi_t)^T \vdots \underline{\Phi}(\psi^*_t)^T\right]^T = \begin{bmatrix} \underline{\Phi}(\psi_t) \\ \cdots \\ \underline{\Phi}(\psi^*_t) \end{bmatrix} \tag{8}$$

Premultiplying (7) by such inverse yields

$$\begin{bmatrix} \underline{\Phi}(\psi_t) \\ \cdots \\ \underline{\Phi}(\psi^*_t) \end{bmatrix} y_t = \begin{bmatrix} \underline{\chi}_t \underline{\Phi}(\phi_t) z_t \\ \cdots\cdots\cdots \\ v^*_t \end{bmatrix} \quad \text{if } L_t > D_t > 0 \tag{9}$$

From (9), it follows that

$$\underline{\Phi}(\psi_t)\mathrm{y}_t = \underline{\chi}_t\underline{\Phi}(\phi_t)z_t \quad \text{if } D_t > 0 \tag{10}$$

Substitution of (4) into (10) yields:

$$\underline{\Phi}(\psi_t)\mathrm{y}_t = \underline{\chi}_t\underline{\Phi}(\phi_t)Hx_t + \underline{\chi}_t\underline{\Phi}(\phi_t)Gv_t \quad \text{if } D_t > 0 \tag{11}$$

Notice that (11) is a linear Gaussian descriptor system [26] with stochastic i.i.d. coefficients $\underline{\Phi}(\psi_t)$ and $\underline{\chi}_t\underline{\Phi}(\phi_t)$. Because χ_t has an inverse, (11) can be transformed into

$$\underline{\chi}_t^T\underline{\Phi}(\psi_t)\mathrm{y}_t = \underline{\Phi}(\phi_t)Hx_t + \underline{\Phi}(\phi_t)Gv_t \quad \text{if } D_t > 0 \tag{12}$$

Next we introduce an auxiliary indicator process $\tilde{\chi}_t$ as follows:

$$\tilde{\chi}_t \triangleq \chi_t^T\Phi(\psi_t) \quad \text{if } D_t > 0.$$

With this we get a simplified version of (12):

$$\underline{\tilde{\chi}}_t\mathrm{y}_t = \underline{\Phi}(\phi_t)Hx_t + \underline{\Phi}(\phi_t)Gv_t \quad \text{if } D_t > 0 \tag{13}$$

Remark 1: For the development of the JPDA, [1, p. 224] makes use of an $L_t \times (M+1)$ dimensional measurement-target association matrix $\Omega_t = [\omega_{ij,t}]$ with $\omega_{ij,t} = 1$ if the ith measurement belongs to target j (with target 0 meaning clutter) and $\omega_{ij,t} = 0$ otherwise. In our setup Ω_t satisfies, for $L_t > 0$:

$$\begin{aligned} \Omega_t &= [\psi_t^* \quad \Phi(\psi_t)^T\chi_t\Phi(\phi_t)], && \text{if } D_t > 0 \\ &= [\psi_t^* \quad \emptyset_{L_t\times M}], && \text{if } D_t = 0 \end{aligned} \tag{14}$$

where $\emptyset_{L_t\times M}$ denotes an $L_t \times M$-dimensional zero-matrix.

IV. Development of the Coupled PDA (CPDA) filter

Let Y_t denote the σ-algebra of measurements y_t up to and including moment t. In this section we develop Bayesian characterizations of the conditional density $p_{x_t|Y_t}(x)$. From (13), it follows that all relevant associations and permutations can be covered by $(\phi_t, \tilde{\chi}_t)$-hypotheses. Hence, through defining the weights

$$\beta_t(\phi,\tilde{\chi}) \triangleq \mathrm{Prob}\{\phi_t = \phi, \tilde{\chi}_t = \tilde{\chi} \mid Y_t\},$$

the law of total probability yields:

$$p_{x_t|Y_t}(x) = \sum_{\tilde{\chi},\phi}\beta_t(\phi,\tilde{\chi})p_{x_t|\phi_t,\tilde{\chi}_t,Y_t}(x \mid \phi,\tilde{\chi}) \tag{15}$$

And thus, our problem is to characterize the terms in the last summation. This problem is solved in two steps, the first of which is the following proposition.

Proposition 1: For any $\phi \in \{0,1\}^M$, such that $D(\phi) \triangleq \sum_{i=1}^M \phi_i \leq L_t$, and any $\tilde{\chi}_t$ matrix realization $\tilde{\chi}$ of size $D(\phi) \times L_t$, the following holds true:

$$p_{x_t|\phi_t,\tilde{\chi}_t,Y_t}(x \mid \phi,\tilde{\chi}) = \frac{p_{\tilde{z}_t|x_t,\phi_t}(\underline{\tilde{\chi}}\mathrm{y}_t \mid x,\phi)\cdot p_{x_t|Y_{t-1}}(x)}{F_t(\phi,\tilde{\chi})} \tag{16}$$

$$\beta_t(\phi,\tilde{\chi}) = F_t(\phi,\tilde{\chi})\lambda^{(L_t-D(\phi))}[\prod_{i=1}^M(1-P_d^i)^{(1-\phi_i)}(P_d^i)^{\phi_i}]/c_t \tag{17}$$

where $\underline{\tilde{\chi}} \triangleq \tilde{\chi}\otimes I_m$, and $F_t(\phi,\tilde{\chi})$ and c_t are such that they normalize $p_{x_t|\phi_t,\tilde{\chi}_t,Y_t}(x \mid \phi,\tilde{\chi})$ and $\beta_t(\phi,\tilde{\chi})$ respectively.

Proof: see Appendix A. The specialty of this proof is because of the derivation of Bayesian equations for the descriptor system (13). ■

Our next step is given by the following Theorem.

Theorem 1: Let $p_{x_t|Y_{t-1}}(x)$ be Gaussian with mean $\bar{x}_t$ and covariance $\bar{P}_t$ and let $F_t(\phi,\tilde{\chi})$ be defined by Proposition 1. Then $F_t(0,\tilde{\chi}) = 1$, whereas for $\phi \neq 0$:

$$\begin{aligned} F_t(\phi,\tilde{\chi}) \quad &= \quad \exp\{-\tfrac{1}{2}\mu_t^T(\phi,\tilde{\chi})Q_t(\phi)^{-1}\mu_t(\phi,\tilde{\chi})\}\cdot \\ &\quad \cdot[(2\pi)^{mD(\phi)}\mathrm{Det}\{Q_t(\phi)\}]^{-\frac{1}{2}} \end{aligned} \tag{18}$$

where

$$\begin{aligned} \mu_t(\phi,\tilde{\chi}) &\triangleq \underline{\tilde{\chi}}\mathrm{y}_t - \underline{\Phi}(\phi)H\bar{x}_t \\ Q_t(\phi) &\triangleq \underline{\Phi}(\phi)(H\bar{P}_tH^T + GG^T)\underline{\Phi}(\phi)^T \end{aligned}$$

Moreover, $p_{x_t|Y_t}(x)$ is a Gaussian mixture, whereas its overall mean $\hat{x}_t$ and its overall covariance $\hat{P}_t$ satisfy

$$\hat{x}_t = \bar{x}_t + \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi)\Big(\sum_{\tilde{\chi}}\beta_t(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi})\Big) \tag{19}$$

$$\begin{aligned} \hat{P}_t = \bar{P}_t &- \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi)\underline{\Phi}(\phi)H\bar{P}_t\Big(\sum_{\tilde{\chi}}\beta_t(\phi,\tilde{\chi})\Big) + \\ &+ \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi)\Big(\sum_{\tilde{\chi}}\beta_t(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi})\mu_t^T(\phi,\tilde{\chi})\Big)K_t^T(\phi) + \\ &- \left(\sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi)\Big(\sum_{\tilde{\chi}}\beta_t(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi})\Big)\right)\cdot \\ &\cdot\left(\sum_{\substack{\phi' \\ \phi'\neq 0}} K_t(\phi')\Big(\sum_{\tilde{\chi}'}\beta_t(\phi',\tilde{\chi}')\mu_t(\phi',\tilde{\chi}')\Big)\right)^T \end{aligned} \tag{20.a}$$

with:

$$\begin{aligned} &K_t(\phi) \triangleq \bar{P}_tH^T\underline{\Phi}(\phi)^TQ_t(\phi)^{-1} \text{ if } \phi \neq 0, \text{ and} \\ &K_t(0) \triangleq 0 \end{aligned} \tag{20.b}$$

Proof: (*Outline*) If $p_{x_t|Y_{t-1}}(x)$ is Gaussian with mean $\bar{x}_t$ and covariance $\bar{P}_t$, then the density $p_{x_t|\phi_t,\tilde{\chi}_t,Y_t}(x \mid \phi,\tilde{\chi})$

is Gaussian with mean $\hat{x}_t(\phi,\tilde{\chi})$ and covariance $\hat{P}_t(\phi)$ satisfying

$$\begin{array}{rlr} \hat{x}_t(\phi,\tilde{\chi}) & = \bar{x}_t + K_t(\phi)[\underline{\tilde{\chi}}\mathrm{y}_t - \underline{\Phi}(\phi)H\bar{x}_t] & \text{if } \phi \neq 0\ , \\ & = \bar{x}_t & \text{if } \phi = 0 \\ \hat{P}_t(\phi) & = \bar{P}_t - K_t(\phi)\underline{\Phi}(\phi)H\bar{P}_t & \text{if } \phi \neq 0\ , \\ & = \bar{P}_t & \text{if } \phi = 0 \end{array}$$

Hence, $p_{x_t|Y_t}(.)$ is a Gaussian mixture, and all equations follow from a lengthy but straightforward evaluation of this mixture. ■

Theorem 1 implies that we get a recursive algorithm if the conditional density $p_{x_t|Y_{t-1}}(x)$, is approximated by a Gaussian shape. We refer to this recursive algorithm as the CPDA filter. It consists of evaluating the following three subsequent steps:

CPDA Step 1 - Prediction:

$$\bar{x}_t = A\hat{x}_{t-1} \qquad (21)$$

$$\bar{P}_t = A\hat{P}_{t-1}A^T + BB^T \qquad (22)$$

CPDA Step 2 - Gating:
For each prediction $(\bar{x}_t^i, \bar{P}_t^i)$, define a gate $G_t^i \in I\!R^m$ as follows:

$$G_t^i \triangleq \{\mathrm{y}' \in I\!R^m; (\mathrm{y}' - h^i\bar{x}_t^i)^T(h^i\bar{P}_t^i h^{iT} + g^i g^{iT})^{-1} \cdot \cdot(\mathrm{y}' - h^i\bar{x}_t^i) \leq \gamma\}$$

with γ the gate size.
If the j-th measurement y_t^j falls outside gate G_t^i, i.e. $\mathrm{y}_t^j \notin G_t^i$, then the i-th component of the j-th column of $\tilde{\chi}_t$ is assumed to equal zero. This reduces the set of possible detection/permutation hypotheses to be evaluated at moment t to, say, $\tilde{\mathcal{X}}_t$.

CPDA Step 3 - Evaluation of the Detection/Permutation Hypotheses, by Using (16) as Approximation:

$$\begin{array}{rll} \beta_t(\phi,\tilde{\chi}) & \cong F_t(\phi\tilde{\chi})\lambda^{(L_t - D(\phi))} \cdot & \\ & \cdot\left[\prod_{i=1}^M (1-P_d^i)^{(1-\phi_i)}(P_d^i)^{\phi_i}\right]/c_t & \text{for } \tilde{\chi} \in \tilde{\mathcal{X}}_t, \\ \beta_t(\phi,\tilde{\chi}) & \cong 0 & \text{else} \end{array} \qquad (23)$$

with $F_t(\phi,\tilde{\chi})$ satisfying equation (18) and c_t a normalizing constant.

CPDA Step 4 - Measurement-Based Update Equations, Using (19) and (20) as Approximations:

$$\hat{x}_t \cong \bar{x}_t + \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi)\left(\sum_{\tilde{\chi}} \beta_t(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi})\right) \qquad (24)$$

$$\begin{aligned} \hat{P}_t \cong \bar{P}_t &- \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi)\underline{\Phi}(\phi)H\bar{P}_t\left(\sum_{\tilde{\chi}} \beta_t(\phi,\tilde{\chi})\right) + \\ &+ \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi)\left(\sum_{\tilde{\chi}} \beta_t(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi})\mu_t^T(\phi,\tilde{\chi})\right)K_t^T(\phi) + \\ &- \left(\sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi)\left(\sum_{\tilde{\chi}} \beta_t(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi})\right)\right) \cdot \\ &\cdot \left(\sum_{\substack{\phi' \\ \phi'\neq 0}} K_t(\phi')\left(\sum_{\tilde{\chi}'} \beta_t(\phi',\tilde{\chi}')\mu_t(\phi',\tilde{\chi}')\right)\right)^T \end{aligned} \qquad (25)$$

with $K_t(\phi)$ defined in (20.b)

Remark 2: Note that the CPDA algorithm has similarities with the JPDA Coupled (JPDAC) filter of [14, ch.6.2, pp. 328-329]: Steps 1 and 3 are equivalent, Step 2 differs slightly only. For the evaluation of Step 4, however, JPDAC uses the additional approximations

$$K_t(\phi) \cong \bar{P}_t H^T(H\bar{P}_t H^T + GG^T)^{-1}\underline{\Phi}(\phi)^T \qquad (26.a)$$

in (24) and in the 2nd and 3rd term of (25), and

$$K_t(\phi)\underline{\Phi}(\phi) \cong \bar{P}_t H^T(H\bar{P}_t H^T + GG^T)^{-1} \qquad (26.b)$$

in the first term of (25).
Strict equality of the latter approximation will hold true in exceptional cases only, e.g., when $\bar{P}_t$ is block-diagonal (i.e., targets far apart), or when $\Phi(\phi) = I$ (i.e., detection probability is unity).

Remark 3: The well-known JPDA filter equations [1] can be obtained from the CPDA equations under the following additional approximate assumption:

$$\hat{P}_t^{ij} \cong 0 \ \text{ for all } i \neq j$$

In addition, CPDA's gating Step 2 simplifies for JPDA, the effect of which is that JPDA's gating approach may eliminate some $(\phi_t,\tilde{\chi}_t)$-hypotheses that are considered by CPDA.

Remark 4: For $M > 1$ the CPDA algorithm obviously is more complex than the JPDA algorithm. If $\lambda = 0$ and $P_d^i = 1$ for all i, and $m \leq n$, then the number of scalar computations for one filter cycle is of the order $\mathcal{O}(M^2n^3 + M^3n^2m + M!M^2m^2)$ for the CPDA and of the order $\mathcal{O}(Mn^3 + Mn^2m + M!M + M^2m^2)$ for the JPDA.

Remark 5: The ENNPDA equations [6] can be obtained from the CPDA equations through pruning all, except the most likely $(\phi,\tilde{\chi})$-hypothesis before executing CPDA step 4, i.e., the weights after pruning become

$$\begin{array}{rll} \beta_t'(\phi,\tilde{\chi}) & = 1 & \text{if } (\phi,\tilde{\chi}) = \underset{(\phi',\tilde{\chi}')}{\text{Argmax}}\ \beta_t(\phi',\tilde{\chi}') \\ & = 0 & \text{else} \end{array}$$

The effect of this dramatic pruning is that the summations in (24) and (25) actually select a single value of ϕ only.

As a result of this, the cross-covariance terms of $\hat{P}_t$ stay zero. The diagonal terms of $\hat{P}_t$ equal the measurement-based update equations of ENNPDA.

V. Track-Coalescence-Avoidance through Hypothesis Pruning

With its ENNPDA approach, [6] has shown that hypothesis pruning can provide an effective approach toward track-coalescence avoidance (see remark 5 of Section IV). The ENNPDA filter equations can be obtained from the CPDA algorithm by pruning all less likely $(\phi_t, \tilde{\chi}_t)$-hypotheses before measurement updating (CPDA step 4). Obviously, ENNPDA's resistance to track coalescence is caused by this pruning, and its sensitivity to missed detections and clutter is also caused by this pruning. Obviously the latter does not occur if $\lambda = 0$ and $P_d^i = 1$ for all i, because in that case $L_t = D_t = M$ and $\Phi(\psi_t) = \Phi(\phi_t) = I_M$. Hence, to reduce the CPDA in that case to the ENNPDA, it is sufficient to prune all less likely χ_t-hypotheses only. Thus, from ENNPDA we know that track-coalescence can be avoided by pruning χ_t-hypotheses. From PDA we know that sensitivity to missed detections and clutter can be avoided by not pruning any ψ_t or ϕ_t hypothesis. Combining these two findings leads to the following new hypothesis pruning strategy: evaluate all (ϕ_t, ψ_t) hypotheses and prune per (ϕ_t, ψ_t)-hypothesis all less-likely χ_t-hypotheses.
To do this, we define for every ϕ and ψ satisfying $D(\psi) = D(\phi) \leq \mathrm{Min}\{M, L_t\}$, a mapping $\hat{\chi}_t(\phi, \psi)$:

$$\hat{\chi}_t(\phi, \psi) \triangleq \underset{\chi}{\mathrm{Argmax}}\ \beta_t\left(\phi, \chi^T \Phi(\psi)\right)$$

where the maximization is over all permutation matrices χ of size $D(\phi) \times D(\phi)$.
The pruning strategy of evaluating all (ϕ, ψ)-hypotheses and only one χ-hypothesis per (ϕ, ψ)-hypothesis implies that we get the following approximated weights $\beta_t'(\phi, \tilde{\chi}) \cong \beta_t(\phi, \tilde{\chi})$:

$$\begin{aligned} \beta_t'\left(\phi, \chi^T \Phi(\psi)\right) &= \beta_t\left(\phi, \chi^T \Phi(\psi)\right) \hat{c}_t \quad \text{if } \chi = \hat{\chi}_t(\phi, \psi) \\ &= 0 \quad \text{else} \end{aligned}$$

with $\hat{c}_t$ a normalization constant for β_t'; i.e.,

$$\sum_{\substack{\phi, \chi, \psi \\ D(\psi) = D(\phi)}} \beta_t'\left(\phi, \chi^T \Phi(\psi)\right) = 1$$

Obviously, this allows for a shorter notation, and we define:

$$\hat{\beta}_t(\phi, \psi) \triangleq \beta_t\left(\phi, \hat{\chi}(\phi, \psi)^T \Phi(\psi)\right) / \hat{c}_t$$

for all (ϕ, ψ) satisfying $D(\phi) = D(\psi) \leq \mathrm{Min}\{M, L_t\}$.
This approach yields a track-coalescence-avoiding Coupled PDA filter, which we refer to as CPDA*.

CPDA Step 1 - Prediction:*
Equivalent to CPDA step 1: (21) and (22)

CPDA Step 2 - Gating:*
Equivalent to CPDA step 2

CPDA Step 3 - Evaluation of the Detection/Permutation Hypotheses:*
Equivalent to CPDA step 3: equations (23) and (18)

CPDA Step 4 - Track-Coalescence Hypothesis Pruning:*
First, evaluate for every (ϕ, ψ) such that $D(\psi) = D(\phi) \leq \mathrm{Min}\{M, L_t\}$:

$$\hat{\chi}_t(\phi, \psi) \triangleq \underset{\chi}{\mathrm{Argmax}}\ \beta_t\left(\phi, \chi^T \Phi(\psi)\right)$$

Next, evaluate all $\hat{\chi}_t(\phi, \psi)$ hypothesis weights

$$\hat{\beta}_t\left(\phi, \psi\right) = \beta_t(\phi, \hat{\chi}_t^T(\phi, \psi)\Phi(\psi))\ / \hat{c}_t$$

where $\hat{c}_t$ is a normalizing constant satisfying

$$\hat{c}_t = \sum_{\phi, \psi} \beta_t\left(\phi, \hat{\chi}_t^T(\phi, \psi)\Phi(\psi)\right)$$

CPDA Step 5 - Measurement-Based Update Equations:*

$$\hat{x}_t \cong \bar{x}_t + \sum_{\substack{\phi \\ \phi \neq 0}} K_t(\phi)\left(\sum_{\substack{\psi \\ D(\psi) = D(\phi)}} \hat{\beta}_t(\phi, \psi)\mu_t\left(\phi, \hat{\chi}_t^T(\phi, \psi)\Phi(\psi)\right)\right) \tag{27}$$

$$\begin{aligned} \hat{P}_t \cong\ & \bar{P}_t - \sum_{\substack{\phi \\ \phi \neq 0}} K_t(\phi)\underline{\Phi}(\phi) H \bar{P}_t\left(\sum_{\substack{\psi \\ D(\psi) = D(\phi)}} \hat{\beta}_t(\phi, \psi)\right) + \\ & + \sum_{\substack{\phi \\ \phi \neq 0}} K_t(\phi)\left(\sum_{\substack{\psi \\ D(\psi) = D(\phi)}} \hat{\beta}_t(\phi, \psi)\mu_t\left(\phi, \hat{\chi}_t^T(\phi, \psi)\Phi(\psi)\right) \cdot \right. \\ & \left. \cdot \mu_t^T\left(\phi, \hat{\chi}_t^T(\phi, \psi)\Phi(\psi)\right)\right) K_t^T(\phi) + \\ & - \left(\sum_{\substack{\phi \\ \phi \neq 0}} K_t(\phi)\left(\sum_{\substack{\psi \\ D(\psi) = D(\phi)}} \hat{\beta}_t(\phi, \psi)\mu_t\left(\phi, \hat{\chi}_t^T(\phi, \psi)\Phi(\psi)\right)\right)\right) \cdot \\ & \cdot \left(\sum_{\substack{\phi' \\ \phi' \neq 0}} K_t(\phi')\left(\sum_{\substack{\psi' \\ D(\psi') = D(\phi')}} \hat{\beta}_t(\phi', \psi')\mu_t\left(\phi', \hat{\chi}_t^T(\phi', \psi')\Phi(\psi')\right)\right)\right)^T \end{aligned} \tag{28}$$

with $\mu_t(.)$ and $K_t(.)$ satisfying equations (18) and (20.b) of CPDA.

The computational complexity of CPDA* is similar to that of CPDA. Hence, our next step is to develop a JPDA* filter from the CPDA* filter in a similar way as the JPDA filter follows from the CPDA filter (see Remark 3). To do so, we first prove the following Theorem:

Theorem 2: Let $p_{x_t|Y_{t-1}}(x)$ be Gaussian with mean $\bar{x}_t = \mathrm{Col}\{\bar{x}_t^1, ..., \bar{x}_t^M\}$ and covariance $\bar{P}_t = \mathrm{Diag}\{\bar{P}_t^1, ..., \bar{P}_t^M\}$, then $\beta_t(\phi, \tilde{\chi})$ of Proposition 1 satisfies:

$$\beta_t(\phi, \tilde{\chi}) = \lambda^{(L_t - D(\phi))}\left[\prod_{i=1}^{M} f_t^i(\phi, \tilde{\chi})(1 - P_d^i)^{(1-\phi_i)}(P_d^i)^{\phi_i}\right] / c_t \tag{29}$$

with:

$$f_t^i(\phi,\tilde{\chi}) = \exp\{-\frac{1}{2}\sum_{k=1}^{L_t}\left([\Phi(\phi)]_{*i}^T\tilde{\chi}_{*k}\mu_t^{ikT}[Q_t^i]^{-1}\mu_t^{ik}\right)\}\cdot$$
$$\cdot[(2\pi)^m \mathrm{Det}\{Q_t^i\}]^{-\frac{1}{2}\phi_i} \quad (30.a)$$

where:

$$\mu_t^{ik} \triangleq \mathrm{y}_t^k - h^i\bar{x}_t^i \quad (30.b)$$
$$Q_t^i \triangleq h^i\bar{P}_t^i h^{iT} + g^i g^{iT} \quad (30.c)$$

whereas $[\Phi(\phi)]_{*i}$ and $\tilde{\chi}_{*i}$ are the ith columns of $\Phi(\phi)$ and $\tilde{\chi}$, respectively. Moreover, $p_{x_t^i|Y_t}(x^i)$, $i \in \{1,...,M\}$, is a Gaussian mixture, while its overall mean $\hat{x}_t^i$ and its overall covariance $\hat{P}_t^i$ satisfy:

$$\hat{x}_t^i = \bar{x}_t^i + W_t^i\left(\sum_{k=1}^{L_t}\beta_t^{ik}\mu_t^{ik}\right) \quad (31.a)$$

$$\hat{P}_t^i = \bar{P}_t^i - W_t^i h^i\bar{P}_t^i\left(\sum_{k=1}^{L_t}\beta_t^{ik}\right) + W_t^i\left(\sum_{k=1}^{L_t}\beta_t^{ik}\mu_t^{ik}\mu_t^{ikT}\right)W_t^{iT} - W_t^i\left(\sum_{k=1}^{L_t}\beta_t^{ik}\mu_t^{ik}\right)\left(\sum_{k=1}^{L_t}\beta_t^{ik}\mu_t^{ik}\right)^T W_t^{iT} \quad (31.b)$$

with:

$$W_t^i \triangleq \bar{P}_t^i h^{iT}[Q_t^i]^{-1}$$
$$\beta_t^{ik} \triangleq \mathrm{Prob}\{[\Phi(\phi)]_{*i,t}^T\tilde{\chi}_{*k,t} = 1 \mid Y_t\} = \sum_{\substack{\phi,\tilde{\chi}\\ \phi\neq 0}}\beta_t(\phi,\tilde{\chi})[\Phi(\phi)]_{*i}^T\tilde{\chi}_{*k}$$

Proof: see Appendix B. The novel part of this proof consists of non-trivial matrix manipulations. Following this, the remaining part is similar to the JPDA derivations and therefore omitted. ■

By combining Theorem 2 with the CPDA* steps, we arrive at the JPDA* filter algorithm.

JPDA Step 1 - Prediction for all* $i \in \{1,...,M\}$:

$$\bar{x}_t^i = a^i\hat{x}_{t-1}^i \quad (32.a)$$
$$\bar{P}_t^i = a^i\hat{P}_{t-1}^i a^{iT} + b^i b^{iT} \quad (32.b)$$

JPDA Step 2 - Gating:*
Equivalent to CPDA* step 2

JPDA Step 3 - Evaluation of the Detection/Evaluation Hypotheses, by Adopting (29) as Approximation:*

$$\beta_t(\phi,\tilde{\chi}) \cong \lambda^{L_t-D(\phi)}[\prod_{i=1}^{M} f_t^i(\phi,\tilde{\chi})(1-P_d^i)^{(1-\phi_i)}(P_d^i)^{\phi_i}]/c_t \quad (33)$$

with $f_t^i(\phi,\tilde{\chi})$, μ_t^{ik} and Q_t^i satisfying (30.a,b,c).

JPDA Step 4 - Track-coalescence hypothesis pruning:*
Equivalent to CPDA* step 4.

JPDA Step 5- Measurement-Based Update Equations for all* $i \in \{1,...,M\}$:

$$\hat{x}_t^i \cong \bar{x}_t^i + W_t^i\left(\sum_{k=1}^{L_t}\hat{\beta}_t^{ik}\mu_t^{ik}\right) \quad (34.a)$$

$$\hat{P}_t^i \cong \bar{P}_t^i - W_t^i h^i\bar{P}_t^i\left(\sum_{k=1}^{L_t}\hat{\beta}_t^{ik}\right) + W_t^i\left(\sum_{k=1}^{L_t}\hat{\beta}_t^{ik}\mu_t^{ik}\mu_t^{ikT}\right)W_t^{iT} - W_t^i\left(\sum_{k=1}^{L_t}\hat{\beta}_t^{ik}\mu_t^{ik}\right)\left(\sum_{k'=1}^{L_t}\hat{\beta}_t^{ik'}\mu_t^{ik'}\right)^T W_t^{iT} \quad (34.b)$$

with

$$W_t^i = \bar{P}_t^i h^{iT}[Q_t^i]^{-1} \quad (34.c)$$
$$\hat{\beta}_t^{ik} = \sum_{\substack{\phi,\psi\\ \phi\neq 0}}\hat{\beta}_t(\phi,\psi)[\Phi(\phi)]_{*i}^T[\hat{\chi}_t^T(\phi,\psi)\Phi(\psi)]_{*k} \quad (34.d)$$

with $[.]_{*k}$ the kth column of [.].

Note that the JPDA* filter simplifies to the well-known JPDA by omitting Step 4 and slightly simplifying the gating mechanism of Step 2. This also implies that the numerical complexity of JPDA* is similar to that of JPDA.

VI. Monte Carlo simulations

In this section, the new filters (CPDA, JPDA* and CPDA*) are evaluated and compared with the existing filters (ENNPDA, JPDA, and JPDAC). In case of a single target, all except the ENNPDA are equal to the ordinary PDA. Obviously, in case of a single target ($M = 1$) and no clutter (i.e., $\lambda = 0$), the ENNPDA is equivalent to the PDA filter as well. In case of multiple targets (i.e., $M > 1$), all six filters differ, unless $\lambda = 0$ or $P_d^i = 1$ for all i. If $P_d^i = 1$, then JPDAC equals CPDA if the same gating is being used, and if both $P_d^i = 1$ and $\lambda = 0$, then JPDA* equals ENNPDA if the same gating is being used. In order to compare the performance of all filters in multiple target situations, Monte Carlo simulations have been performed. In order to simplify the comparisons for all filters, the same gating procedure is used (CPDA Step 2).

The simulations we used are based on a simple crossing target scenario, similar to the one used by [6]: two targets modeled as constant velocity objects that move towards each other with a given relative velocity V_{rel}, cross at a certain moment in time and then move away from each other with the same relative velocity. Each simulation that starts with perfect estimates is run for 50 scans, with the crossover point at scan 10.

For each target, the underlying model of the potential target measurements is given by (1) and (3)

$$x_{t+1}^i = a^i x_t^i + b^i w_t^i \quad (1)$$
$$z_t^i = h^i x_t^i + g^i v_t^i \quad (3)$$

Furthermore, for $i = 1, 2$:

$$a^i = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \qquad b^i = \sigma_a^i \cdot \begin{bmatrix} \frac{1}{2}T_s^2 \\ T_s \end{bmatrix},$$

$$h^i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad g^i = \sigma_m^i$$

where σ_a^i represents the standard deviation of acceleration noise and σ_m^i represents the standard deviation of the measurement error. For simplicity we consider the situation of similar targets only; i.e., $\sigma_a^i = \sigma_a$, $\sigma_m^i = \sigma_m$, $P_d^i = P_d$.

Following [6] and [27], we define the tracking index $\Lambda \triangleq T_s^2 \sigma_a / \sigma_m$ and the normalized relative velocity $V_{\text{rel}}^{\text{norm}} = V_{\text{rel}} T_s / \sigma_m$. With this, the scenario parameters are $V_{\text{rel}}^{\text{norm}}$, Λ, T_s, σ_m, λ, P_d and the gate size γ. Table I gives the scenario parameter values that are being used.

TABLE I
Scenario parameter values.

Scenario	P_d	λ	σ_m	Λ	γ	T_s	$V_{\text{rel}}^{\text{norm}}$
1	1	0	30	$3\frac{1}{3}$	25	10	Variable
2	0.9	0	30	$3\frac{1}{3}$	25	10	Variable
3	1	0.001	30	$3\frac{1}{3}$	25	10	Variable
4	0.9	0.001	30	$3\frac{1}{3}$	25	10	Variable
5	0.9	0.001	30	1	25	10	Variable

During our simulations we counted track i "O.K." if

$$| h^i \hat{x}_T^i - h^i x_T^i | \leq 9\sigma_m$$

and we counted track $i \neq j$ "Swapped" if

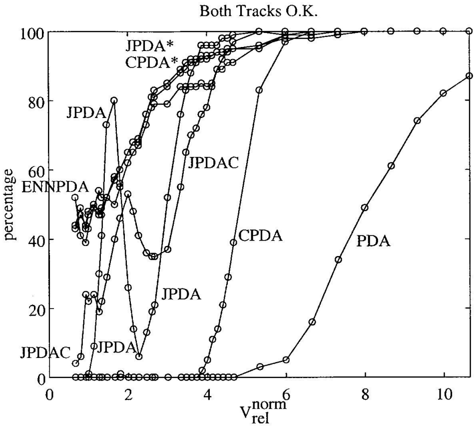
$$| h^i \hat{x}_T^i - h^j x_T^j | \leq 9\sigma_m$$

Furthermore, two tracks $i \neq j$ are counted "Coalescing" at scan t, if

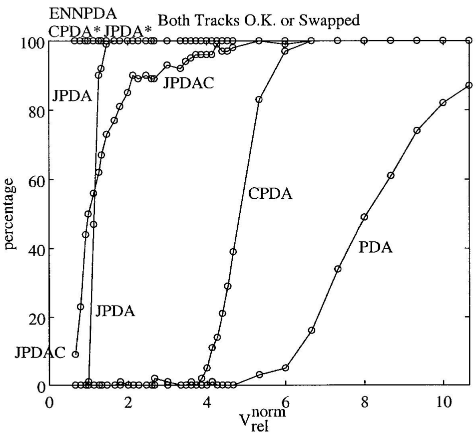
$$| h^i \hat{x}_t^i - h^j \hat{x}_t^j | \leq \sigma_m \wedge | h^i x_t^i - h^j x_t^j | > \sigma_m$$

For each of the scenarios in table 1, Monte Carlo simulations containing 100 runs of the crossing trajectories have been performed for each of the tracking filters. To make the comparisons more meaningful, for all tracking mechanisms the same random number streams were used. To rule out possible CPDA covariance matrix singularities, such as reported by [24], the simulations were performed in double precision and it was verified that the ratio between largest and smallest eigenvalues of the covariance matrices stayed low enough.

For scenario 1, results similar to [6] were obtained. Results of the Monte Carlo simulations for the other scenarios



1a. Both tracks "O.K." percentage



1b. Both tracks "O.K." or "Swapped" percentage

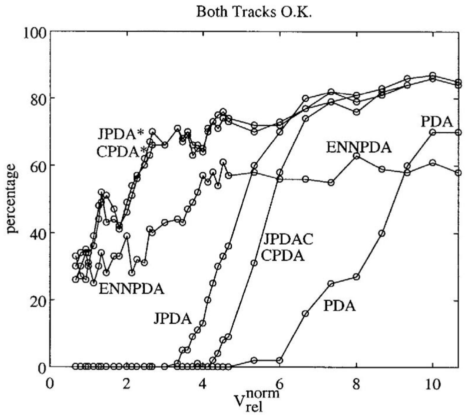
Fig. 1. Simulation results for scenario 2, with $P_d = 0.9$, $\lambda = 0$ and $\Lambda = 3\frac{1}{3}$.

are depicted as function of the normalized relative velocities in three types of figures, showing respectively:

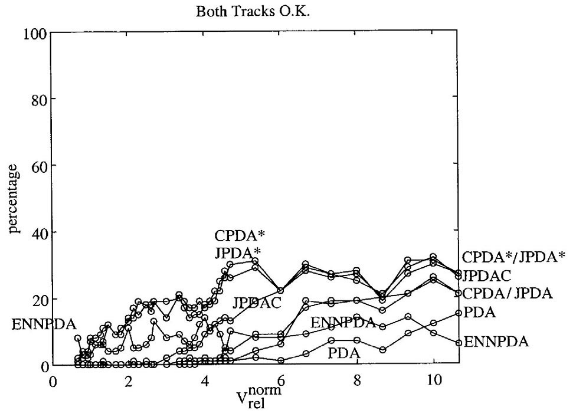
- The percentage of Both tracks "O.K. " (figures 1a,2a,3a,4a).
- The percentage of Both tracks "O.K." or "Swapped" (figures 1b,2b,3b,4b).
- The average number of "Coalescing" scans (figures 5a,5b).

For the scenarios considered, the simulation results show a superior performance of the JPDA* and CPDA* filters above the other filters. They also show that CPDA performs less than JPDAC, which on its turn may be competitive with JPDA. Scenario 4 results show that there are also cases in which none of the algorithms considered perform really satisfactorily, although JPDA* and CPDA* perform best. In addition to this, more detailed observations have been made for the following comparisons:

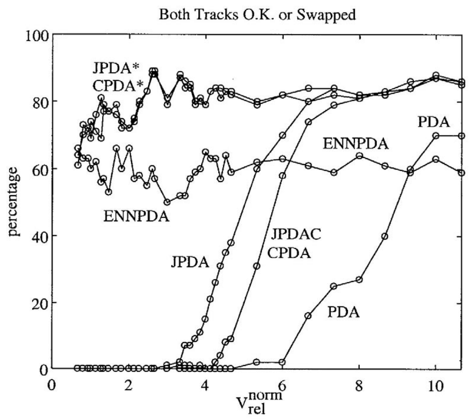
- JPDA outperforms CPDA,
- JPDAC outperforms CPDA if $P_d < 1$,



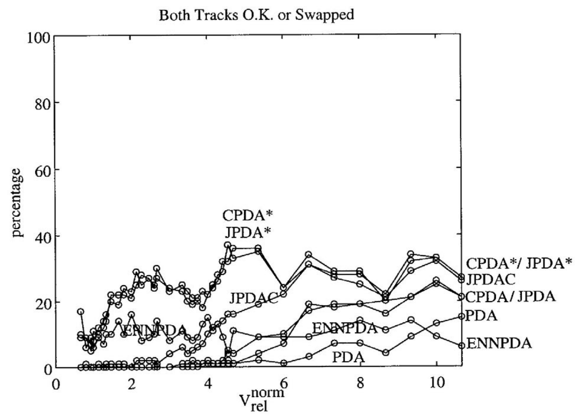
2a. Both Tracks “O.K.” percentage



3a. Both Tracks “O.K.” percentage



2b. Both Tracks “O.K.” or “Swapped” percentage



3b. Both Tracks “O.K.” or “Swapped” percentage

Fig. 2. Simulation results for scenario 3, with $P_d = 1$, $\lambda = 0.001$ and $\Lambda = 3\frac{1}{3}$.

Fig. 3. Simulation results for scenario 4, with $P_d = 0.9$, $\lambda = 0.001$ and $\Lambda = 3\frac{1}{3}$.

- JPDA* outperforms JPDA and JPDAC,
- JPDA* outperforms ENNPDA,
- CPDA* may perform marginally better than JPDA*.

JPDA outperforms CPDA

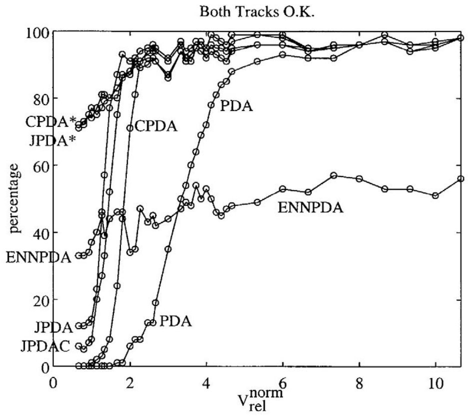
CPDA appears to be a case where the optimal Gaussian approximation of the exact Bayesian filter equations for the conditional density leads to worse performance than a non-optimal Gaussian approximation of JPDA. The explanation for this phenomenon is that the conditional density for the joint state of slowly crossing targets has a multi-modality of a particular form: apart from the target identity the joint state is almost known. For slowly crossing targets this may result into a strong coupling between the two tracks through the cross-covariance terms, which supports CPDA's strong preference for keeping both tracks in between competing measurements. Because of JPDA's negligence of these cross-covariance terms, the latter effect is less strong for JPDA. This explains why CPDA degrades in performance at significantly higher relative velocities than JPDA.

JPDAC outperforms CPDA if $P_d < 1$

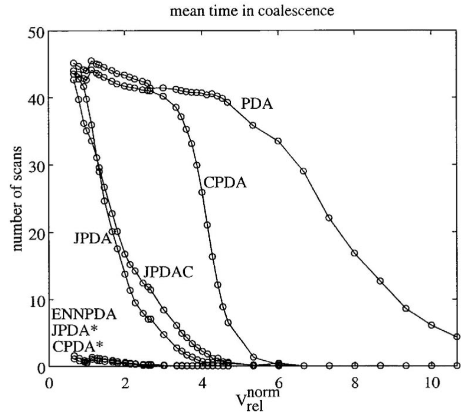
If $P_d = 1$, JPDAC and CPDA perform equally. For $P_d = 0.9$, however, JPDAC outperforms CPDA. The explanation is that because of the approximation adopted, JPDAC's filter gain (26) varies significantly less than CPDA's filter gain (20.b) if $P_d < 1$. As a consequence, JPDAC tends to neglect cross-covariance terms during the measurement-based track update, in case of missed detections. As such JPDAC becomes competitive with JPDA if $P_d < 1$.

JPDA* outperforms JPDA and JPDAC

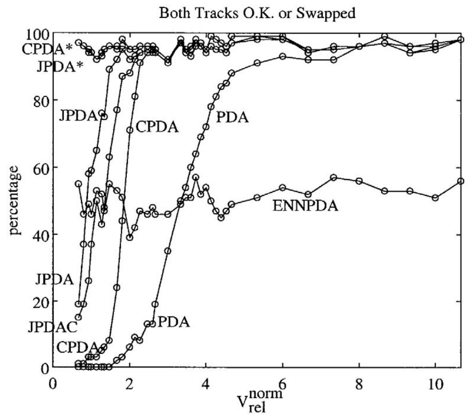
For the examples considered, JPDA* clearly retains JPDA's insensitivity to clutter and missed detections for the full range of relative velocities. At the same time, JPDA* clearly outperforms JPDA and JPDAC when the relative velocities become small enough. This indeed ap-



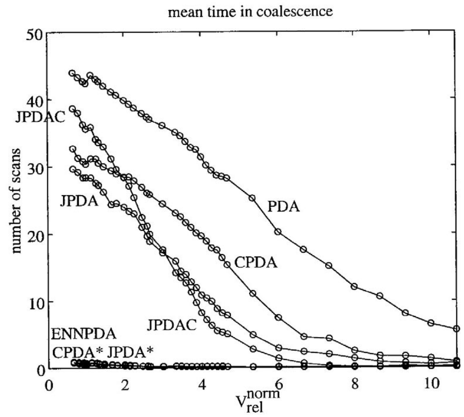
4a. Both Tracks "O.K." percentage



5a. Scenario 2, $P_d = 0.9$, $\lambda = 0$ and $\Lambda = 3\frac{1}{3}$



4b. Both Tracks "O.K." or "Swapped" percentage



5b. Scenario 4, $P_d = 0.9$, $\lambda = 0.001$ and $\Lambda = 3\frac{1}{3}$

Fig. 4. Simulation results for scenario 5, with $P_d = 0.9$, $\lambda = 0.001$ and $\Lambda = 1$.

Fig. 5. Typical results in terms of average number of "coalescing" scans

pears to be caused by JPDA*'s avoidance of track coalescence. Instead of coalescing tracks, JPDA* either performs O.K. or may swap tracks. For a better understanding of this difference, consider the situation $\lambda = 0$ and $P_d = 1$ (scenario 1). Then at each scan only two hypotheses exist with non-zero probability. JPDA and JPDAC use both hypotheses for track updating, while JPDA* uses the most likely hypothesis only for track updating. If the probabilities of the two hypotheses become almost the same and the measurements are clearly separated (which is very likely at low relative velocities), then JPDA and JPDAC tend to update both tracks somewhere in between the two measurements, and JPDA* updates the tracks at separate positions as indicated by the two separated measurements. In this simple example, JPDA and JPDAC tend to coalesce both tracks, and JPDA* has a probability of about 50% both tracks "O.K", about 50% to swap both tracks and no track coalescence or track loss. Similar findings also apply to scenarios 2-5.

JPDA* outperforms ENNPDA

If $P_d = 1$ and $\lambda = 0$, then JPDA* and ENNPDA perform equally well. For $\lambda = 0.001$, however, JPDA* performs significantly better than ENNPDA. Track coalescence is not observed. If the detection probability $P_d < 1$ or $\lambda > 0$ then JPDA* clearly outperforms ENNPDA. The differences also appear when the relative velocities are large. This simply illustrates that JPDA* indeed avoids ENNPDA's sensitivity to clutter and missed detections.

CPDA* may perform marginally better than JPDA*

CPDA* and JPDA* hardly show difference in performance. For small relative velocities only, CPDA* may perform only marginally better than JPDA*. Thus in this case, memorizing cross-covariance between crossing tracks does neither lead to a worse performance nor to a really improved performance. It means that because of the track-coalescence-avoiding hypothesis pruning method of Section V, the practical use of memorizing cross-covariance between crossing tracks seems to disappear.

VII. Concluding remarks

In this paper, new directions in PDA development have been explored. First, in sections II and III, the multi-target tracking problem has been embedded into a problem of filtering for a linear descriptor system with stochastic i.i.d. coefficients. Subsequently, in section IV, the exact Bayesian and Gaussian approximated filter equations have been developed. The resulting filter algorithm has been named CPDA, and appeared to differ significantly from the JPDA filter and, if detection probability is not unity, also from the more recent JPDAC filter. Then, in section V, an hypothesis pruning strategy has been developed that allows both to avoid JPDA's sensitivity to track coalescence, and to preserve JPDA's insensitivity to clutter and missed detections. Application of this new pruning strategy to JPDA and CPDA resulted into two new algorithms that have been named JPDA* and CPDA*, respectively. Finally, in section VI, the new developments have been evaluated through Monte Carlo simulations for some characteristic multitarget tracking scenarios. Both JPDA and JPDAC appeared to outperform CPDA. On their turn, however, JPDA* and CPDA* appeared to outperform them all, simply due to their ENNPDA inherited insensitivity to track coalescence. Moreover, JPDA* and CPDA* appeared to perform similarly.

On the basis of the results obtained, it is also possible to draw some more general conclusions. First, the commonly established approach of approximating exact Bayesian filter equations, by assuming a centred Gaussian approximation for the conditional density, appears to lead to a filter (CPDA) that performs less good than those based on other Gaussian approximations (i.e. JPDA, JPDAC, JPDA* and CPDA*). In order to identify a logical explanation for this phenomenon, we notice that the conditional density of the targets' joint state has a particular multi-modality: in addition to the local optimum for the non-swapped tracks, often other local optima exist for track swap possibilities. The approach of centering a Gaussian optimally (in MMSE sense) between these local optima implies a preference to track coalescence over track swap. For tracking applications, however, it is better to accurately know the target locations while being uncertain about the track identities because of possible track swap, than to know the identities and be uncertain about the target locations because of possible track coalescence; thus, track swap is preferred over track coalescence. This preference implies that a straightforward application of an MMSE sense optimality criterion is not so practical when it comes to tracking closely spaced targets. Obviously, ENNPDA, JPDA* and CPDA* avoid track coalescence through centering a Gaussian density around one of the local optima (which need not be the global optimum), which appears to be so effective that the need for remembering the coupling between tracks practically disappears. The optimal centering of JPDA, JPDAC and CPDA in MMSE sense, simply prefers track coalescence over track swap, which is a less good choice from a practical tracking point of view.

Having identified the elementary practical problem that comes with adopting an MMSE optimality criterion, it is interesting to see what this practically means for Kastella's MLE, for Kamen's SME, and for Pao's MTMR:

- For the SME approach of Kamen, it is clear that due to the one-to-one SME transformation of the measurement equation [19]-[21], the data association problem seems totally avoided. Since this SME transformation is one-to-one and it does not change the targets' joint state space, it has no influence at all on the exact conditional density. In addition to its transformation, the SME approach uses the EKF approach towards approximately evaluating the conditional density. Since an EKF is based on optimality in MMSE sense, one may expect that Kamen's SME filter will prefer track coalescence above track swap, similarly to JPDA. The latter agrees well with the findings in [20] that an SME filter performs similarly to a JPDA filter, thus preferring track coalescence over track swap.
- Kastella's MLE approach tries to center a Gaussian density around the global optimum. As such, we should expect that MLE will neither swap nor coalesce tracks as long as the local optima, representing track swap possibilities, stay sufficiently apart. During the period that those local optima do not stay sufficiently apart, however, the MLE approach also prefers track coalescence over track swap. The memorization during such period of the coupling (nonzero cross-covariance terms) between individual target tracks, however, allows Kastella's MLE approach to react very effectively as soon as targets split. These considerations correspond with the practically sound results reported for Kastella's MLE approach [25].
- With the MTMR approach [12], the conditional density is approximated by a sum of Gaussian densities, the covariances of which are not coupled, however. From a theoretical point of view, the latter seems an omission. On the basis of our experience that CPDA* performs almost equally well as JPDA*, however, we may expect that as long as MTMR retains a Gaussian for each individual target and each relevant track swap possibility, then MTMR's performance should not suffer from neglecting the coupling between individual Gaussians. When, however, the targets stay sufficiently long and close enough to each other, then the latter condition is not satisfied and MTMR might tend to centre for each target a single Gaussian in between the local optima, which implies track coalescence. Thus, when targets cross at low relative velocities, Pao's MTMR is expected to perform similarly to JPDA.

Finally, the newly developed embedding of multitarget tracking into filtering for a linear descriptor system with stochastic coefficients, has shown to be of value in the derivation of exact Bayesian and Gaussian approximate multitarget tracking equations. This embedding for example enabled us to avoid the heuristic reasoning such as necessary at the time of JPDAC development. As such, this new embedding formulates multitarget tracking problems within the modeling framework of nonlinear filtering theory, which eventually may lead to the exploration of further developments in PDA, e.g., incorporating the crucial plot resolution problem [28], [29]. Similarly, the new representation might support developments in effectively combining multi-target probabilistic data association approaches with other Bayesian solutions, e.g. with IMM [16], [30], [31], with imaging sensor models [15], [32]. Presently, at NLR good progress is made in effectively incorporating IMM, plot resolution model, and imaging sensor models, the results of which have already been added to those of [31] during the realization of Eurocontrol's multitarget multisensor tracking system ARTAS.

APPENDIX A Proof of Proposition 1

If $\phi = 0$ we get $p_{x_t|\phi_t,\tilde{\chi}_t,Y_t}(x \mid 0, \tilde{\chi}) = p_{x_t|Y_{t-1}}(x)$. Else

$$\begin{aligned} & p_{x_t|\phi_t,\tilde{\chi}_t,Y_t}(x \mid \phi, \tilde{\chi}) = \\ & \quad = p_{x_t|\phi_t,\tilde{\chi}_t,\mathrm{y}_t,L_t,Y_{t-1}}(x \mid \phi, \tilde{\chi}, \mathrm{y}_t, L_t) = \\ & \quad = p_{x_t|\phi_t,\tilde{\chi}_t,\mathrm{y}_t,L_t,\tilde{z}_t,Y_{t-1}}(x \mid \phi, \tilde{\chi}, \mathrm{y}_t, L_t, \underline{\tilde{\chi}}\mathrm{y}_t) = \\ & \quad = p_{x_t|\phi_t,\tilde{z}_t,Y_{t-1}}(x \mid \phi, \underline{\tilde{\chi}}\mathrm{y}_t) = \\ & \quad = p_{\tilde{z}_t|x_t,\phi_t}(\underline{\tilde{\chi}}\mathrm{y}_t \mid x, \phi) p_{x_t|Y_{t-1}}(x) / F_t(\phi, \tilde{\chi}) \end{aligned}$$

with $F_t(\phi, \tilde{\chi}) \triangleq p_{\tilde{z}_t|\phi_t,Y_{t-1}}(\underline{\tilde{\chi}}\mathrm{y}_t \mid \phi)$. Subsequently

$$\begin{aligned} & \beta_t(\phi, \tilde{\chi}) \triangleq \mathrm{Prob}\{\phi_t = \phi, \tilde{\chi}_t = \tilde{\chi} \mid Y_t\} = \\ & \quad = p_{\phi_t,\tilde{\chi}_t|Y_t}(\phi, \tilde{\chi}) = \\ & \quad = p_{\phi_t,\tilde{\chi}_t|\mathrm{y}_t,L_t,Y_{t-1}}(\phi, \tilde{\chi} \mid \mathrm{y}_t, L_t) = \\ & \quad = p_{\mathrm{y}_t,\tilde{\chi}_t|\phi_t,L_t,Y_{t-1}}(\mathrm{y}_t, \tilde{\chi} \mid \phi, L_t) p_{\phi_t|L_t,Y_{t-1}}(\phi \mid L_t) / c'_t \end{aligned}$$

If $D_t > 0$ we have

$$\begin{aligned} \tilde{\chi}_t^T \tilde{\chi}_t &= \Phi(\psi_t)^T \chi_t \chi_t^T \Phi(\psi_t) = \\ &= \Phi(\psi_t)^T \Phi(\psi_t) = \\ &= \mathrm{Diag}\{\psi_t\} \end{aligned}$$

$$\begin{aligned} \tilde{\chi}_t \Phi(\psi_t)^T &= \chi_t^T \Phi(\psi_t) \Phi(\psi_t)^T = \\ &= \chi_t^T \end{aligned}$$

which means that the transformation from (ψ_t, χ_t) into $\tilde{\chi}_t$ has an inverse which implies

$$\begin{aligned} & p_{\mathrm{y}_t,\tilde{\chi}_t|\phi_t,L_t,Y_{t-1}}(\mathrm{y}_t, \chi^T \Phi(\psi) \mid \phi, L_t) = \\ & \quad = p_{\mathrm{y}_t,\psi_t,\chi_t|\phi_t,L_t,Y_{t-1}}(\mathrm{y}_t, \psi, \chi \mid \phi, L_t) \end{aligned}$$

Furthermore, because the transformation from $(\mathrm{y}_t, \psi_t, \chi_t)$ into $(\tilde{z}_t, v_t^*, \psi_t, \chi_t)$ is a permutation, we get for $L_t > D(\phi) > 0$

$$\begin{aligned} & p_{\mathrm{y}_t,\psi_t,\chi_t|\phi_t,L_t,Y_{t-1}}(\mathrm{y}_t, \psi, \chi \mid \phi, L_t) = \\ & \quad = p_{\tilde{z}_t,v_t^*,\psi_t,\chi_t|\phi_t,L_t,Y_{t-1}}(\underline{\chi}^T \underline{\Phi}(\psi)\mathrm{y}_t, \underline{\Phi}(\psi^*)\mathrm{y}_t, \psi, \chi \mid \phi, L_t) \end{aligned}$$

Hence, for $L_t > D(\phi) > 0$, β_t satisfies:

$$\begin{aligned} & \beta_t(\phi, \chi^T \Phi(\psi)) = F_t(\phi, \chi^T \Phi(\psi)) p_{v_t^*|\phi_t,L_t}(\underline{\Phi}(\psi^*)\mathrm{y}_t \mid \phi, L_t) \cdot \\ & \quad p_{\psi_t|\phi_t,L_t}(\psi \mid \phi, L_t) p_{\chi_t|\phi_t}(\chi \mid \phi) p_{L_t|\phi_t}(L_t \mid \phi) p_{\phi_t}(\phi) / c'' \end{aligned}$$

Subsequently using the JPDA derivation [1] yields:

$$\begin{aligned} & \beta_t(\phi, \chi^T \Phi(\psi)) = F_t(\phi, \chi^T \Phi(\psi)) \lambda^{(L_t - D(\phi))} \cdot \\ & \quad \cdot [\prod_{i=1}^{M} (P_d^i)^{\phi_i} (1 - P_d^i)^{(1-\phi_i)}] / c_t \end{aligned}$$

with c_t a normalizing constant. It can be easily verified that the last equation also holds true if $L_t = D(\phi)$ or $D(\phi) = 0$. ■

APPENDIX B Proof of Theorem 2

Because $\bar{P}_t$ is block-diagonal, $(H\bar{P}_t H^T + GG^T)$ and $\underline{\Phi}(\phi)(H\bar{P}_t H^T + GG^T)\underline{\Phi}(\phi)^T$ are block-diagonal too. Hence, it can be shown that

$$\begin{aligned} & \underline{\Phi}(\phi)^T \left(\underline{\Phi}(\phi)(H\bar{P}_t H^T + GG^T)\underline{\Phi}(\phi)^T \right)^{-1} = \\ & \quad = \underline{\Phi}(\phi)^T \underline{\Phi}(\phi)(H\bar{P}_t H^T + GG^T)^{-1} \underline{\Phi}(\phi)^T \end{aligned}$$

Because of the form of $\Phi(.)$, we know $\Phi(\phi)^T \Phi(\phi) = \mathrm{Diag}\{\phi_1, ..., \phi_M\}$. Hence, the former simplifies to

$$\begin{aligned} & \underline{\Phi}(\phi)^T \left(\underline{\Phi}(\phi)(H\bar{P}_t H^T + GG^T)\underline{\Phi}(\phi)^T \right)^{-1} = \\ & \quad = (H\bar{P}_t H^T + GG^T)^{-1} \underline{\Phi}(\phi)^T \end{aligned}$$

Because $Q_t(\phi) = \underline{\Phi}(\phi)(H\bar{P}_t H^T + GG^T)\underline{\Phi}(\phi)^T$ is block-diagonal, we get

$$\begin{aligned} \mathrm{Det}\{Q_t(\phi)\} &= \mathrm{Det}\{\underline{\Phi}(\phi)(H\bar{P}_t H^T + GG^T)\underline{\Phi}(\phi)^T\} = \\ &= \prod_{\substack{i=1 \\ \phi_i \neq 0}}^{M} \mathrm{Det}\{h^i \bar{P}_t^i h^{iT} + g^i g^{iT}\} = \\ &= \prod_{i=1}^{M} \left(\mathrm{Det}\{Q_t^i\} \right)^{\phi_i} \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} & \mu_t(\phi, \tilde{\chi})^T Q_t(\phi)^{-1} \mu_t(\phi, \tilde{\chi}) = \\ & \quad = \mu_t(\phi, \tilde{\chi})^T \left(\underline{\Phi}(\phi)(H\bar{P}_t H^T + GG^T)\underline{\Phi}(\phi)^T \right)^{-1} \mu_t(\phi, \tilde{\chi}) = \\ & \quad = \mu_t(\phi, \tilde{\chi})^T \underline{\Phi}(\phi) \left(H\bar{P}_t H^T + GG^T \right)^{-1} \underline{\Phi}(\phi)^T \mu_t(\phi, \tilde{\chi}) = \\ & \quad = \sum_{i=1}^{M} \left(\left(\mu_t(\phi, \tilde{\chi})^T \underline{\Phi}(\phi) \right)^i \left(h^i \bar{P}_t^i h^{iT} + g^i g^{iT} \right)^{-1} \cdot \right. \\ & \qquad \qquad \left. \cdot \left(\underline{\Phi}(\phi)^T \mu_t(\phi, \tilde{\chi}) \right)^i \right) = \\ & \quad = \sum_{i=1}^{M} \left(\underline{\Phi}(\phi)^T \mu_t(\phi, \tilde{\chi}) \right)^{iT} \left(Q_t^i \right)^{-1} \left(\underline{\Phi}(\phi)^T \mu_t(\phi, \tilde{\chi}) \right)^i \stackrel{*}{=} \\ & \quad = \sum_{i=1}^{M} \left(\left(\sum_{k=1}^{L_t} [\Phi(\phi)]_{*i}^T \tilde{\chi}_{*k} \mu_t^{ik} \right)^T \left(Q_t^i \right)^{-1} \cdot \right. \\ & \qquad \qquad \left. \cdot \left(\sum_{k'=1}^{L_t} [\Phi(\phi)]_{*i}^T \tilde{\chi}_{*k'} \mu_t^{ik'} \right) \right) = \\ & \quad = \sum_{i=1}^{M} \sum_{k=1}^{L_t} \left(\sum_{k'=1}^{L_t} [\Phi(\phi)]_{*i}^T \tilde{\chi}_{*k} [\Phi(\phi)]_{*i}^T \cdot \right. \end{aligned}$$

$$\cdot\tilde{\chi}_{*k'}\left(\mu_t^{ik}\right)^T\left(Q_t^i\right)^{-1}\mu_t^{ik'}\Big) =$$

$$= \sum_{i=1}^{M}\sum_{k=1}^{L_t}[\Phi(\phi)]^T_{*i}\tilde{\chi}_{*k}[\Phi(\phi)]^T_{*i}\tilde{\chi}_{*k}\left(\mu_t^{ik}\right)^T\left(Q_t^i\right)^{-1}\mu_t^{ik} =$$

$$= \sum_{i=1}^{M}\sum_{k=1}^{L_t}[\Phi(\phi)]^T_{*i}\tilde{\chi}_{*k}\left(\mu_t^{ik}\right)^T\left(Q_t^i\right)^{-1}\mu^{ik} \qquad \text{(B.2)}$$

where for $\stackrel{*}{=}$ use is made of

$$\left(\underline{\Phi}(\phi)^T\mu_t(\phi,\tilde{\chi})\right)^i =$$

$$= \left(\underline{\Phi}(\phi)^T\underline{\tilde{\chi}}\mathrm{y}_t - \underline{\Phi}(\phi)^T\underline{\Phi}(\phi)H\bar{x}_t\right)^i =$$

$$= \sum_{k=1}^{L_t}\left(\left(\Phi(\phi)^T\tilde{\chi}\right)^{ik}\mathrm{y}_t^k\right) - \left(\Phi(\phi)^T\Phi(\phi)\right)^{ii}h^i\bar{x}_t^i =$$

$$= \sum_{k=1}^{L_t}\Big(\left(\Phi(\phi)^T\tilde{\chi}\right)^{ik}\mathrm{y}_t^k - \left(\Phi(\phi)^T\tilde{\chi}\right)^{ik}\cdot$$

$$\cdot\left(\Phi(\phi)^T\Phi(\phi)\right)^{ii}h^i\bar{x}_t^i\Big) =$$

$$= \sum_{k=1}^{L_t}\left(\left(\Phi(\phi)^T\tilde{\chi}\right)^{ik}\mathrm{y}_t^k - \left(\Phi(\phi)^T\tilde{\chi}\right)^{ik}h^i\bar{x}_t^i\right) =$$

$$= \sum_{k=1}^{L_t}\left(\Phi(\phi)^T\tilde{\chi}\right)^{ik}\mu_t^{ik} =$$

$$= \sum_{k=1}^{L_t}[\Phi(\phi)]^T_{*i}\tilde{\chi}_{*k}\mu_t^{ik}$$

Substituting (B.1) and (B.2) into (18) yields

$$F_t(\phi,\tilde{\chi}) = \prod_{i=1}^{M} f_t^i(\phi,\tilde{\chi}) \qquad \text{(B.3)}$$

with $f_t^i(\phi,\tilde{\chi})$ as given by (30). Substituting (B.3) into (16) yields (29), and (31.a) and (31.b) follow as for JPDA [1]. ■

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