# Tracking Multiple Maneuvering Targets From Possibly Unresolved, Missing or False Measurements 

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## Summary

In Daum (1992) it has been very well explained that sensor resolution modeling is crucial for the tracking of closely spaced targets. For non-maneuvering targets appropriate models and tracking approaches have been developed by Chang \& Bar-Shalom (1984) and by Koch \& Van Keuk (1997). This paper combines their sensor resolution submodels with a descriptor system approach towards tracking two suddenly maneuvering closely spaced targets from measurements that may be false, missing or unresolved. Using this descriptor system formalism, exact Bayesian and approximate filter equations are derived.

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## 1 Introduction

In [1] and [2] it has been explained that sensor resolution modeling is very important for the tracking of closely spaced targets. The key issue is that the probability of resolution typically is worse than the probability of correct measurement association. Hence, optimizing the latter requires modeling the former. In literature there are a few papers which develop good resolution models and incorporate them into effective target tracking. [3] introduces a hard measurement distance threshold model regarding yes/no resolution, and incorporates the corresponding Error density function within JPDA. [4] incorporates this Error function modeling with Multiple Hypothesis Tracking (MHT). [5] introduces a Gaussian shaped measure for the probability of resolution, and shows that this combines smoothly with a Gaussian mixture framework of MHT. [6] enhances this MHT approach towards suddenly maneuvering targets by incorporating IMM. [7] utilizes group tracking to determine false and missing measurements and assess targets that merge with or split from a group.
The aim of this paper is to combine resolution submodels of [3] and [5] with the descriptor system approach by the authors towards tracking multiple maneuvering targets from missing and false measurements. This approach was introduced in [8] to derive, in a mathematically unambiguous way, exact and approximate Bayesian equations for filtering multiple targets from possibly missing and false measurements. For linear Gaussian targets the descriptor system approach did lead to novel tracking filters which were referred to as CPDA, CPDA* and JPDA*, where the * refers to a particular track coalescence avoiding pruning of permutation hypotheses [8]. Of these filters, CPDA performs comparable to JPDA [9] and JPDA-Coupled [10], whereas CPDA* and JPDA* performs significantly better. In a follow-up series of studies the descriptor system approach is extended to situations of Markov jump linear targets, including the development of several novel approximate Bayesian filters, i.e.:

- IMMJPDA* [11], [12] which prunes particular permutation hypotheses (similar as JPDA*) into a descriptor system version of the IMMJPDA filter by [13];
- JIMMCPDA [14] which is in theory the best combination of PDA and IMM for multiple Markov jump linear targets. ${ }^{1}$
- JIMMCPDA* [15] which prunes particular permutation hypotheses in JIMMCPDA similar as in CPDA* ;
- Particle filter implementations of the exact Bayesian filter equations [14], [16].

[^0]A comparison of these tracking filters through Monte Carlo simulations [14] shows that JIMMCPDA* and IMMJPDA* performs much better than JIMMCPDA and IMMJPDA, and also remarkably well in comparison to a good particle filter implementation of the exact Bayesian filter equations.
This paper is organized as follows. Section 2 defines the two-target tracking problem considered. Section 3 formulates this as a problem of filtering for a jump-linear descriptor system with independent identically distributed (i.i.d.) stochastic coefficients. Section 4 develops exact Bayesian filter equations. Section 5 develops equations for the conditional mean and covariance. Sections 6 and 7 specify the JIMMCPDAR and JIMMCPDAR* filters respectively. Section 8 presents a discussion of the results.

## 2 The two-target tracking problem

We consider two targets and assume that the state of each target is modeled as a jump linear system:

$$
\begin{equation*}
x_{t+1}^{i}=a^{i}\left(\theta_{t+1}^{i}\right) x_{t}^{i}+b^{i}\left(\theta_{t+1}^{i}\right) w_{t}^{i}, \quad i=1,2 \tag{1}
\end{equation*}
$$

where $x_{t}^{i}$ is the $n$-vectorial state of the $i$-th target, $\theta_{t}^{i}$ is the Markovian switching mode of the $i$-th target and assumes values from $\mathrm{M} \square\{1, . ., N\}$ according to a transition probability matrix $\Pi^{i}, a^{i}\left(\theta_{t}^{i}\right)$ and $b^{i}\left(\theta_{t}^{i}\right)$ are $(n \times n)$ - and $\left(n \times n^{\prime}\right)$-matrices and $w_{t}^{i}$ is a sequence of i.i.d. standard Gaussian variables of dimension $n^{\prime}$ with $w_{t}^{i}, w_{t}^{j}$ independent for all $i \neq j$ and $w_{t}^{i}, x_{0}^{i}, x_{0}^{j}$ independent for all $i \neq j$.

We assume that a potential measurement originating from target $i$ is also modeled as a jump linear system:

$$
\begin{equation*}
z_{t}^{i}=h^{i}\left(\theta_{t}^{i}\right) x_{t}^{i}+g^{i}\left(\theta_{t}^{i}\right) v_{t}^{i}, \quad i=1,2 \tag{2}
\end{equation*}
$$

where $z_{t}^{i}$ is an $m$-vector, $h^{i}\left(\theta_{t}^{i}\right)$ is an $(m \times n)$-matrix and $g^{i}\left(\theta_{t}^{i}\right)$ is an $\left(m \times m^{\prime}\right)$-matrix, and $v_{t}^{i}$ is a sequence of i.i.d. standard Gaussian variables of dimension $m^{\prime}$ with $v_{t}^{i}$ and $v_{t}^{j}$ independent for all $i \neq j$. Moreover $v_{t}^{i}$ is independent of $x_{0}^{j}$ and $w_{t}^{j}$ for all $i, j$.
Let $\quad x_{t}=\left[\begin{array}{c}x_{t}^{1} \\ x_{t}^{2}\end{array}\right], \theta_{t}=\left[\begin{array}{c}\theta_{t}^{1} \\ \theta_{t}^{2}\end{array}\right], w_{t}=\left[\begin{array}{c}w_{t}^{1} \\ w_{t}^{2}\end{array}\right]$, and $A\left(\theta_{t}\right)=\operatorname{Diag}\left\{a^{1}\left(\theta_{t}^{1}\right), a^{2}\left(\theta_{t}^{2}\right)\right\}, B\left(\theta_{t}\right)=\operatorname{Diag}\left\{b^{1}\left(\theta_{t}^{1}\right), b^{2}\left(\theta_{t}^{2}\right)\right\}$,
Then we can model the state of our 2 targets as follows:

$$
\begin{equation*}
x_{t+1}=A\left(\theta_{t+1}\right) x_{t}+B\left(\theta_{t+1}\right) w_{t} \tag{3}
\end{equation*}
$$

with $A$ and $B$ of size $2 n \times 2 n$ and $2 n \times 2 n^{\prime}$ respectively, with $\left\{\theta_{t}\right\}$ assuming values from $\{1, \ldots, N\}^{2}$ according to transition probability matrix $\Pi=\left[\Pi_{\eta, \theta}\right]$. If the 2 targets switch mode independently of each other, then:

$$
\begin{equation*}
\Pi_{\eta, \theta}=\prod_{i=1}^{2} \Pi_{\eta^{\prime}, \theta^{i}}^{i} \tag{4}
\end{equation*}
$$

for every $\eta$ and $\theta \in \mathrm{M}^{2}=\{1, \ldots, N\}^{2}$.

Next with $z_{t}=\left[\begin{array}{l}z_{t}^{1} \\ z_{t}^{2}\end{array}\right], v_{t}=\left[\begin{array}{l}v_{t}^{1} \\ v_{t}^{2}\end{array}\right]$, and $H\left(\theta_{t}\right)=\operatorname{Diag}\left\{h^{1}\left(\theta_{t}^{1}\right), h^{2}\left(\theta_{t}^{2}\right)\right\}, G\left(\theta_{t}\right)=\operatorname{Diag}\left\{g^{1}\left(\theta_{t}^{1}\right), g^{2}\left(\theta_{t}^{2}\right)\right\}$, we obtain:

$$
\begin{equation*}
z_{t}=H\left(\theta_{t}\right) x_{t}+G\left(\theta_{t}\right) v_{t} \tag{5}
\end{equation*}
$$

with $H\left(\theta_{t}\right)$ and $G\left(\theta_{t}\right)$ of size $2 m \times 2 n$ and $2 m \times 2 m^{\prime}$ respectively.
In order to incorporate limited sensor resolution with Bayesian filtering, [5] adopts a non-zero probability that two close targets merge. This event of merging or not is represented by a zero-one-valued process $\kappa_{t}$, where $\kappa_{t}=1$ refers to merging, and $\kappa_{t}=0$ means non-merging.

Following [5] we assume:

$$
\begin{align*}
& p_{\kappa_{i} \mid x_{t}, \theta_{t}}(0 \mid x, \theta)=  \tag{6}\\
& \begin{array}{rl}
p_{\kappa_{t} \mid x_{t}, \theta_{t}} & 1-p_{\kappa_{i} \mid x_{t}, \theta_{t}}(1 \mid x, \theta)= \\
& \exp \left\{-\frac{1}{2}\left(h^{1}\left(\theta^{1}\right) x^{1}-h^{2}\left(\theta^{2}\right) x^{2}\right)^{T} R(\theta)^{-1} \cdot\left(h^{1}\left(\theta^{1}\right) x^{1}-h^{2}\left(\theta^{2}\right) x^{2}\right)\right\}= \\
& =\exp \left\{-\frac{1}{2} x^{T} H(\theta)^{T}\left[\begin{array}{c}
I \\
-I
\end{array}\right] R(\theta)^{-1}[I \vdots-I] H(\theta) x\right\}
\end{array}
\end{align*}
$$

where $R(\theta)$ is an $m \times m$ resolution capability matrix:

$$
\begin{align*}
R(\theta) & =\left(g^{1}\left(\theta^{1}\right) I_{r} g^{1}\left(\theta^{1}\right)^{T}+g^{2}\left(\theta^{2}\right) I_{r} g^{2}\left(\theta^{2}\right)^{T}\right)= \\
& =[I \vdots I] G(\theta) \operatorname{Diag}\left\{I_{r}, I_{r}\right\} G(\theta)^{T}[I \vdots I]^{T} \tag{8}
\end{align*}
$$

with $I_{r}=\operatorname{Diag}\left\{r_{1}, \ldots, r_{m}\right\}$ and typically $r_{i}>1$.
Following [3], [10], for $\kappa_{t}=1$ we assume that with probability $P_{d}^{0}$ the merged potential measurement $\left(z_{t}^{1}+z_{t}^{2}\right) / 2$ is observed at moment $t$. And for $\kappa_{t}=0$, we assume that with a nonzero detection probability, $P_{d}^{i}$, the potential measurement $z_{t}^{i}$ is observed at moment $t$, independently per target.
Let $F_{t}$ denote the number of false measurements at moment $t$, we assume $F_{t}$ to be Poisson distributed:

$$
\begin{align*}
p_{F_{t}}(F) & =\frac{(\lambda V)^{F}}{F!} \exp (-\lambda V), \quad F=0,1,2, \ldots  \tag{9.a}\\
& =0,
\end{align*}
$$

where $\lambda$ is the spatial density of false measurements and $V$ is the volume of the observed region. Thus $\lambda V$ is the expected number of false measurements in the observed region. We
assume that the false measurements are uniformly distributed in the observed region, which means that a column-vector $v_{t}^{*}$ of $F_{t}$ i.i.d. false measurements has the following density:

$$
\begin{equation*}
p_{v_{i}^{\prime} F_{t}}\left(v^{*} \mid F\right)=V^{-F} \tag{9.b}
\end{equation*}
$$

Furthermore we assume that the process $\left\{v_{t}^{*}\right\}$ is a sequence of independent vectors, which are independent of $\left\{x_{t}\right\},\left\{w_{t}\right\},\left\{v_{t}\right\}$ and of the merging and detection.
At moment $t$ a vector observation $y_{t}$ is made, the components of which consist of $F_{t}$ false measurements and $D_{t}$ detected (merged) potential measurements, in an arbitrary order. The total number $L_{t}$ of measurements is:

$$
\begin{equation*}
L_{t}=D_{t}+F_{t} . \tag{9.c}
\end{equation*}
$$

The multi-target tracking problem is to estimate $\left(x_{t}, \theta_{t}\right)$ from observations $\left.Y_{t}{ }^{\Delta}{ }^{\Delta} L_{s}, y_{s} ; 0 \leq s \leq t\right\}$.

## 3 Descriptor system formulation

In order to prepare for a Bayesian evaluation of this tracking problem, in this section we develop an equation which relates the potential measurements $z_{t}$ mathematically to the true measurements $y_{t}$.
Let $\phi_{i, t}$ be the detection indicator of target $i$. For $\kappa_{t}=0$ it assumes the value 1 with probability $P_{d}^{i}>0$, independently of $\phi_{j, t}, j \neq i$, and the value 0 with probability $\left(1-P_{d}^{i}\right)$. For $\kappa_{t}=1$, with probability $P_{d}^{0}>0$ the two potential measurements merge to form one true measurement, i.e. $\phi_{i, t}=\frac{1}{2}$ for $i=1,2$, and with probability $1-P_{d}^{0}$ the merged potential measurements do not form a true measurement, i.e. $\phi_{i, t}=0$ for $i=1,2$. The resulting detection indicator vector $\phi_{t}=\left[\phi_{1, t} \phi_{2, t}\right]^{T}$ is a sequence $\left\{\phi_{t}\right\}$ of i.i.d. vectors, and with a $\kappa_{t}$-conditional distribution:

$$
\begin{align*}
p_{\phi \mid, k_{\mathrm{c}}}(\phi \mid \kappa) & =\prod_{i=1}^{2}\left(1-P_{\mathrm{d}}^{i}\right)^{1-\phi}\left(P_{\mathrm{d}}^{i}\right)^{\phi} \text { if } \kappa=0, \phi \in\{0,1\}^{2} \\
& =\prod_{i=1}^{2}\left(1-P_{d}^{0}\right)^{\frac{1}{2}-\phi}\left(P_{d}^{0}\right)^{\phi} \text { if } \kappa=1, \phi \in\left\{\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}\right]\right\}  \tag{10}\\
& =0 \quad \text { else. }
\end{align*}
$$

Summation over all components of $\phi_{t}$ yields

$$
\begin{equation*}
D_{t}=\sum_{i=1}^{M} \phi_{i, t} . \tag{11}
\end{equation*}
$$

We want to use this target indicator process $\left\{\phi_{t}\right\}$ within an equation that relates $y_{t}$ to $z_{t}$. To accomplish this, we first introduce the following operator $\Phi$ : for an $M^{\prime}$-vector $\phi^{\prime}$ with ( 0,1 )valued components $\phi_{i}^{\prime}, \quad i \in\left\{1, M^{\prime}\right\}$, we define $D\left(\phi^{\prime}\right)=\triangleq \sum_{i=1}^{M^{\prime}} \phi_{i}^{\prime}$ and the operator $\Phi$ producing $\Phi\left(\phi^{\prime}\right)$ as a $(0,1)$-valued matrix of size $D\left(\phi^{\prime}\right) \times M^{\prime}$ of which the $i$ th row equals the $i$ th non-zero row of $\operatorname{Diag}\left\{\phi^{\prime}\right\}$. And, if $\phi^{\prime}=\left[\frac{1}{2} \frac{1}{2}\right]^{T}$, then $\Phi\left(\phi^{\prime}\right)=\left[\frac{1}{2} \frac{1}{2}\right]$. We also define an underlining notation: $\underline{\Phi}\left(\phi^{\prime}\right){ }^{\Delta} \Phi\left(\phi^{\prime}\right) \otimes I_{m}$, with $I_{m}$ a unit-matrix of size $m$ and $\otimes$ Kronecker product, i.e.

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \otimes I_{m}=\left[\begin{array}{ccc}
a I_{m} & \vdots & b I_{m} \\
& \ldots \ldots . . \\
c I_{m} & \vdots & d I_{m}
\end{array}\right]
$$

With this the vector of detected measurements $\tilde{z}_{t}$ satisfies:

$$
\begin{equation*}
\tilde{z}_{t}=\Phi\left(\phi_{t}\right) z_{t} \quad \text { if } D_{t}>0 \tag{12}
\end{equation*}
$$

We next introduce a stochastic $D_{t} \times L_{t}$-matrix process $\left\{\tilde{\chi}_{t}\right\}$ such that

$$
\begin{equation*}
\tilde{\tilde{\chi}}_{t} y_{t}=\tilde{z}_{t} \quad \text { if } D_{t}>0 \tag{13}
\end{equation*}
$$

where $\underline{\chi}_{t} \stackrel{\Delta}{=} \tilde{\chi}_{t} \otimes I_{m}$.
Substitution of (5) into (12) and this into (13) yield:

$$
\begin{equation*}
\underline{\tilde{\chi}}_{t} y_{t}=\underline{\Phi}\left(\phi_{t}\right) H\left(\theta_{t}\right) x_{t}+\underline{\Phi}\left(\phi_{t}\right) G\left(\theta_{t}\right) v_{t} \quad \text { if } \quad D_{t}>0 \tag{14}
\end{equation*}
$$

Notice that the size of $\underline{\tilde{\chi}}_{t}$ is $D_{t} m \times L_{t} m$ and the size of $\underline{\Phi}\left(\phi_{t}\right)$ is $D_{t} m \times M m$. Equation (14) is a jump-linear Gaussian descriptor system [18] with stochastic i.i.d. coefficients $\underline{\tilde{\chi}}_{t}$ and $\underline{\Phi}\left(\phi_{t}\right)$. Together with equations (3), (4) and (6) through (11), equation (14) captures the filtering problem to be solved in a mathematically well defined system of equations.
Following (14), all target to measurement relevant associations and permutations are covered by $\left(\phi_{t}, \tilde{\chi}_{t}\right)$-hypotheses with $D_{t}>0$. To this set of hypotheses we add one for the situation $D_{t}=0$ through the hypothesis $\phi_{t}=\{0\}^{2}$ and $\tilde{\chi}_{t}=\{ \}^{L_{t}}$. Hence, through defining the weights

$$
\begin{equation*}
\beta_{t}(\phi, \kappa, \tilde{\chi}, \theta)^{\stackrel{\Delta}{=}} \operatorname{Prob}\left\{\phi_{t}=\phi, \kappa_{t}=\kappa, \tilde{\chi}_{t}=\tilde{\chi}, \theta_{t}=\theta \mid Y_{t}\right\} \tag{15}
\end{equation*}
$$

the law of total probability yields:

$$
\begin{equation*}
p_{x_{i}, \theta_{t} \mid T_{t}}(x, \theta)=\sum_{\tilde{\chi}, \phi, \kappa} \beta_{t}(\phi, \kappa, \tilde{\chi}, \theta) p_{x_{i} \mid \theta_{t}, \phi_{t}, \kappa_{t}, \tilde{\chi}_{t} Y_{t}}(x \mid \theta, \phi, \kappa, \tilde{\chi}) \tag{16}
\end{equation*}
$$

With this the problem is to characterize the terms in the last summation, which is done in the next section.

## 4 Bayesian filter equations

In this section a Bayesian characterization of the conditional density $p_{x_{i}, \theta_{t} \bar{Y}_{t}}(x, \theta)$ is given where $Y_{t}$ denotes the $\sigma$-algebra generated by measurements up to and including moment $t$.

## Proposition 1

For any $\phi \in\{0,1\}^{2} \cup\left\{\left[\frac{1}{2} \frac{1}{2}\right]^{T}\right\}$, such that $D(\phi)=\sum_{i=1}^{\Delta} \phi_{i} \leq L_{t}$, and any $\tilde{\chi}_{t}$ matrix realization $\tilde{\chi}$ of size $D(\phi) \times L_{t}$, the following holds true:

$$
\begin{align*}
& p_{x_{i} \mid \theta_{t}, \kappa_{t}, \phi_{t}, \tilde{\chi}_{t} Y_{t}}(x \mid \theta, \kappa, \phi, \tilde{\chi})=\frac{p_{\tilde{z}_{t} \mid x_{t}, \theta_{t}, \phi_{t}}\left(\underline{\chi} y_{t} \mid x, \theta, \phi\right) \cdot p_{x_{i} \mid \theta_{t}, \kappa_{t}, Y_{t-1}}(x \mid \theta, \kappa)}{F_{t}(\phi, \kappa, \tilde{\chi}, \theta)}  \tag{17}\\
& \beta_{t}(\phi, \kappa, \tilde{\chi}, \theta)=F_{t}(\phi, \kappa, \tilde{\chi}, \theta) \lambda^{\left(L_{t}-D(\phi)\right)} \cdot p_{\phi_{t} \mid \kappa_{t}}(\phi \mid \kappa) \cdot p_{\kappa_{t} \mid \theta_{t}, Y_{t-1}}(\kappa \mid \theta) \cdot p_{\theta_{t} \mid Y_{t-1}}(\theta) / c_{t} \tag{18}
\end{align*}
$$

where $F_{t}(\phi, \kappa, \tilde{\chi}, \theta)$ and $c_{t}$ are such that they normalize $p_{x_{i} \theta_{t}, \kappa_{t}, \phi_{t} \tilde{\tilde{x}}_{t} Y_{t}}(x \mid \theta, \kappa, \phi, \tilde{\chi})$ and $\beta_{t}(\phi, \kappa, \tilde{\chi}, \theta)$ respectively.

Proof: Omitted due to space limitation. It largely follows the proof of Theorem 1 in [16]

## Proposition 2

The conditional density $p_{x_{t} \mid \theta_{t}, \kappa_{t}, Y_{t-1}}(x \mid \theta, \kappa)$ in Proposition 1 satisfies for $\kappa_{t}=0$ and $\kappa_{t}=1$ respectively:

$$
\begin{align*}
& p_{x_{i} \mid \theta_{t}, \kappa_{i}, Y_{t-1}}(x \mid \theta, 0)=\frac{1}{1-c_{t}(\theta)} p_{x_{i} \mid \theta_{t}, Y_{t-1}}(x \mid \theta)-\frac{c_{t}(\theta)}{1-c_{t}(\theta)} p_{x_{i} \mid \theta_{t}, \kappa_{t}, Y_{t-1}}(x \mid \theta, 1)  \tag{19}\\
& p_{x_{i} \mid \theta_{t}, \kappa_{t}, Y_{t-1}}(x \mid \theta, 1)=p_{\kappa_{i} \mid x_{t}, \theta_{t}}(1 \mid x, \theta) p_{x_{i} \mid \theta_{t}, Y_{t-1}}(x \mid \theta) / c_{t}(\theta)  \tag{20}\\
& c_{t}(\theta)=p_{\kappa_{i} \mid \theta_{t}, Y_{t-1}}(1 \mid \theta) \tag{21}
\end{align*}
$$

Proof: Follows from Bayes and subsequent evaluation.
The next step is to combine Propositions 1 and 2 for the derivation of a characterization of the exact Bayesian measurement update equations in the following Theorem.

## Theorem 1

The measurement updating of $p_{x_{t}, \theta_{t} \mid Y_{t-1}}(x, \theta)$ to $p_{x_{x_{t}}, \theta_{i} Y_{t}}(x, \theta)$ satisfies, for $\theta \in\{1, . ., N\}^{2}$ :

$$
\begin{align*}
& p_{x_{i}, \theta_{i} \mid Y_{t}}(x, \theta)=\sum_{\phi, \tilde{\chi}} \beta_{t}^{0}(\theta, \tilde{\chi}, \phi) p_{x_{i} \mid \theta_{t}, \tilde{y}_{t}, \phi_{t}, Y_{t}}^{r=0}(x \mid \theta, \tilde{\chi}, \phi)+\sum_{\phi, \tilde{\chi}} \beta_{t}^{1}(\theta, \tilde{\chi}, \phi) p_{x_{i} \mid \theta_{t}, \kappa_{t}, \tilde{\chi}_{t}, \phi_{i} Y_{t}}(x \mid \theta, 1, \tilde{\chi}, \phi)  \tag{22}\\
& \beta_{t}^{\kappa}(\theta, \tilde{\chi}, \phi)=\frac{1}{c_{t}} F_{t}^{\kappa}(\theta, \tilde{\chi}, \phi) c_{t}(\theta)^{\kappa} \lambda^{(2-D(\phi))} \cdot\left[p_{\phi_{t} \mid K_{t}}(\phi \mid \kappa)-\kappa p_{\phi_{t} \mid K_{t}}(\phi \mid 0)\right] p_{\theta_{t} \mid Y_{t-1}}(\theta)  \tag{23}\\
& p_{x_{i} \mid \theta_{t}, \tilde{t}_{t}, \phi_{t}, Y_{t}}^{r=0}(x \mid \theta, \tilde{\chi}, \phi)=\frac{p_{\tilde{z}_{t} \mid x_{t}, \theta_{t}, \phi_{t}}\left(\underline{\tilde{\chi}} y_{t} \mid x, \theta, \phi\right) p_{x_{i} \mid \theta_{t}, Y_{t-1}}(x \mid \theta)}{F_{t}^{0}(\theta, \tilde{\chi}, \phi)}  \tag{24}\\
& p_{x_{i} \mid \theta_{t}, \kappa_{t}, \tilde{\chi}_{t}, \phi_{t}, Y_{t}}(x \mid \theta, 1, \tilde{\chi}, \phi)=\frac{p_{\tilde{z}_{t} \mid x_{t}, \theta_{t}, \phi_{t}}\left(\underline{\tilde{\chi}} y_{t} \mid x, \theta, \phi\right) p_{x_{i} \mid \theta_{t}, \kappa_{t}, Y_{t-1}}(x \mid \theta, 1)}{F_{t}^{1}(\theta, \tilde{\chi}, \phi)}  \tag{25}\\
& p_{\tilde{z}_{t} \mid x_{t}, \theta_{t}, \phi_{t}}\left(\underline{\tilde{\chi}} y_{t} \mid x, \theta, \phi\right)=N\left\{\underline{\tilde{\chi}} y_{t} ; \underline{\Phi}(\phi) H(\theta) x, \underline{\Phi}(\phi) G(\theta) G(\theta)^{T} \underline{\Phi}(\phi)^{T}\right\} \tag{26}
\end{align*}
$$

with $p_{x_{t} \mid \theta_{t}, \kappa_{t}, Y_{t-1}}(x \mid \theta, 1)$ and $c_{t}(\theta)$ satisfying (20) and (21) respectively, $F_{t}^{0}(\theta, \tilde{\chi}, \phi)$ and $F_{t}^{1}(\theta, \tilde{\chi}, \phi)$ normalization functions, and $c_{t}$ such that $\sum_{\theta, \kappa, \phi, \tilde{\chi}} \beta_{t}^{\kappa}(\theta, \chi, \phi)=1$.

## Proof of Theorem 1

When $r=0$ there is always resolution, which implies eq. (24). Under the condition $r>0$ we substitute (20) into (17) for $\kappa=1$ to get equation (25) with $F_{t}^{1}(\theta, \tilde{\chi}, \phi)=F_{t}(\theta, 1, \tilde{\chi}, \phi)$.
To get (22) and (23), we first substitute (19) into (17) for $\kappa=0$ :

$$
p_{x_{t} \mid \theta_{t}, \kappa_{t}, \tilde{\chi}_{t}, \phi_{t}, Y_{t}}(x \mid \theta, 0, \tilde{\chi}, \phi)=p_{\tilde{z}_{t} \mid x_{t}, \theta_{t}, \phi_{t}}\left(\underline{\tilde{\chi}} y_{t} \mid x, \theta, \phi\right) \cdot\left[\frac{p_{x_{i} \mid \theta_{\theta}, Y_{t-1}}(x \mid \theta)}{\left[1-c_{t}(\theta)\right] F_{t}(\theta, 0, \tilde{\chi}, \phi)}-\frac{c_{t}(\theta) p_{x_{1} \mid \theta_{\theta}, \kappa_{t}, Y_{t-1}}(x \mid \theta, 1)}{\left[1-c_{t}(\theta)\right] F_{t}(\theta, 0, \tilde{\chi}, \phi)}\right] \text { (27) }
$$

Next, substituting (24) and (25) into (27) and subsequent evaluation yields:

$$
\begin{align*}
& p_{x_{t} \mid \theta_{0}, \kappa_{t}, \tilde{x}_{i}, \phi_{,}, Y_{t}}(x \mid \theta, 0, \tilde{\chi}, \phi)= \\
& =\frac{F_{t}^{0}(\theta, \tilde{\chi}, \phi)}{F_{t}(\theta, 0, \tilde{\chi}, \phi)} p_{x_{i} \theta_{t}, \tilde{\chi}_{t}, \phi_{t} Y_{t}}^{r=0}(x \mid \theta, \tilde{\chi}, \phi) /\left(1-c_{t}(\theta)\right)-c_{t}(\theta) \frac{F_{t}^{1}(\theta, \tilde{\chi}, \phi)}{F_{t}(\theta, 0, \tilde{\chi}, \phi)} p_{x_{i} \mid \theta_{t}, \kappa_{t}, \tilde{\chi}_{t}, \phi_{t}, Y_{t}}(x \mid \theta, 1, \tilde{\chi}, \phi) /\left(1-c_{t}(\theta)\right)= \\
& =\frac{1}{1-c_{t}^{1}(\theta, \phi, \tilde{\chi})} p_{x_{t} \mid \theta_{t}, \tilde{x}_{t}, \phi_{t}, Y_{t}}^{r=0}(x \mid \theta, \tilde{\chi}, \phi)-\frac{c_{t}^{1}(\theta, \phi, \tilde{\chi})}{1-c_{t}^{1}(\theta, \phi, \tilde{\chi})} p_{x_{t} \mid \theta_{t}, \kappa_{t}, \tilde{\chi}_{t}, \phi_{t}, Y_{t}}(x \mid \theta, 1, \tilde{\chi}, \phi) \tag{H1}
\end{align*}
$$

with:

$$
\begin{align*}
& c_{t}^{1}(\theta, \tilde{\chi}, \phi)=\frac{F_{t}^{1}(\theta, \tilde{\chi}, \phi)}{F_{t}^{0}(\theta, \tilde{\chi}, \phi)} c_{t}(\theta), \text { and }  \tag{H2}\\
& F_{t}(\theta, 0, \tilde{\chi}, \phi)=\frac{F_{t}^{0}(\theta, \tilde{\chi}, \phi)}{\left(1-c_{t}(\theta)\right)}-c_{t}(\theta) \frac{F_{t}^{1}(\theta, \tilde{\chi}, \phi)}{\left(1-c_{t}(\theta)\right)} \tag{H3}
\end{align*}
$$

Substitution of equation (H1) into equation (16) yields:

$$
\begin{aligned}
p_{x_{t}, \theta_{t} \mid Y_{t}}(x \mid \theta)=\sum_{\phi, \tilde{\chi}} \beta_{t}(\theta, 1, \tilde{\chi}, \phi) p_{x_{i} \mid \theta_{t}, \kappa_{t}, \tilde{\chi}_{t}, \phi_{t}, Y_{t}}(x \mid \theta, 1, \tilde{\chi}, \phi)+\sum_{\phi, \tilde{\chi}} \frac{\beta_{t}(\theta, 0, \tilde{\chi}, \phi)}{\left(1-c_{t}^{1}(\theta, \phi, \tilde{\chi})\right)} p_{x_{1}=0}^{r=0} \theta_{t}, \tilde{\chi}_{t}, \phi_{t} Y_{t}
\end{aligned}(x \mid \theta, \tilde{\chi}, \phi)+,
$$

which implies (22) with:

$$
\begin{aligned}
& \beta_{t}^{0}(\theta, \tilde{\chi}, \phi)=\frac{\beta_{t}(\theta, 0, \tilde{\chi}, \phi)}{\left(1-c_{t}^{1}(\theta, \phi, \tilde{\chi})\right)}, \text { and } \\
& \beta_{t}^{1}(\theta, \tilde{\chi}, \phi)=\beta_{t}(\theta, 1, \tilde{\chi}, \phi)-\beta_{t}(\theta, 0, \tilde{\chi}, \phi) \frac{c_{t}^{1}(\theta, \phi, \tilde{\chi})}{\left(1-c_{t}^{1}(\theta, \phi, \tilde{\chi})\right)}
\end{aligned}
$$

Substitution of (18) and subsequent evaluation, using (H2) and (H3), yield:

$$
\begin{aligned}
& \beta_{t}^{0}(\theta, \tilde{\chi}, \phi)=\frac{1}{c_{t}} F_{t}^{0}(\theta, \tilde{\chi}, \phi) \lambda^{(2-D(\phi))} p_{\phi_{t} \mid k_{t}}(\phi \mid 0) p_{\theta_{t} \mid Y_{t-1}}(\theta) \\
& \begin{aligned}
\beta_{t}^{1}(\theta, \tilde{\chi}, \phi) & =\frac{1}{c_{t}} F_{t}^{1}(\theta, \tilde{\chi}, \phi) c_{t}(\theta) \lambda^{(2-D(\phi))} p_{\phi_{t} \mid k_{t}}(\phi \mid 1) p_{\theta_{t} \mid Y_{t-1}}(\theta)-\beta_{t}^{0}(\theta, \tilde{\chi}, \phi) c_{t}^{1}(\theta, \tilde{\chi}, \phi)= \\
& =\frac{1}{c_{t}} F_{t}^{1}(\theta, \tilde{\chi}, \phi) c_{t}(\theta) \lambda^{(2-D(\phi))} p_{\phi_{i} \mid k_{t}}(\phi \mid 1) p_{\theta_{t} \mid Y_{t-1}}(\theta)-\frac{1}{c_{t}} F_{t}^{1}(\theta, \tilde{\chi}, \phi) c_{t}(\theta) \lambda^{(2-D(\phi))} p_{\phi_{i} \mid k_{t}}(\phi \mid 0) p_{\theta_{t} \mid Y_{t-1}}(\theta)= \\
& =\frac{1}{c_{t}} F_{t}^{1}(\theta, \tilde{\chi}, \phi) c_{t}(\theta) \lambda^{(2-D(\phi))}\left[p_{\phi_{i} \mid k_{t}}(\phi \mid 1)-p_{\phi_{\mid} \mid k_{t}}(\phi \mid 0)\right] p_{\theta_{1} \mid Y_{t-1}}(\theta)
\end{aligned}
\end{aligned}
$$

which implies eq. (23). From the above also follows $\sum_{\kappa=0}^{1} \beta_{t}^{\kappa}(\theta, \tilde{\chi}, \phi)=\sum_{\kappa=0}^{1} \beta_{t}(\theta, \kappa, \tilde{\chi}, \phi)$. Hence $c_{t}$ normalizes not only $\beta_{t}(\theta, \kappa, \tilde{\chi}, \phi)$ but also $\beta_{t}^{\kappa}(\theta, \tilde{\chi}, \phi)$.
Q.E.D.

## 5 Conditional mean and covariance

Having characterized a set of equations for the measurement update of the conditional density, our next step is to assume that the joint-mode-conditionally predicted joint target state has a density $p_{x_{t} \mid \theta_{t}, Y_{t-1}}(x \mid \theta)$ which is Gaussian for each joint $\theta$.

## Theorem 2

For each $\theta \in\{1, \ldots, N\}^{2}$, let $p_{x_{t} \mid \theta_{t} Y_{t-1}}(x \mid \theta)$ be Gaussian with mean $\bar{x}_{t}(\theta)$, covariance $\bar{P}_{t}(\theta)$ and let $p_{\theta_{t} \mid Y_{t-1}}(\theta)>0$. Then $p_{x_{1} \mid \theta_{t}, Y_{t}}(x \mid \theta)$ is a Gaussian mixture, with overall weight $p_{\theta_{t} \mid Y_{t}}(\theta)$, overall mean $\hat{x}_{t}(\theta)$ and overall covariance $\hat{P}_{t}(\theta)$, satisfying:

$$
\begin{equation*}
p_{\theta_{t} \mid Y_{t}}(\theta)=\sum_{\kappa, \phi, \tilde{\chi}} \beta_{t}^{\kappa}(\phi, \tilde{\chi}, \theta) \tag{28}
\end{equation*}
$$

where $\beta_{t}^{\kappa}(\phi, \tilde{\chi}, \theta)$ satisfies (25), and:

$$
\begin{align*}
\hat{x}_{t}(\theta)= & \sum_{\phi, \tilde{\chi}} \beta_{t \mid \theta}^{1}(\tilde{\chi}, \phi) \hat{x}_{t}^{1}(\theta, \tilde{\chi}, \phi)+\sum_{\phi, \tilde{\chi}} \beta_{t \mid \theta}^{0}(\tilde{\chi}, \phi) \hat{x}_{t}^{0}(\theta, \tilde{\chi}, \phi)  \tag{29}\\
\hat{P}_{t}(\theta)= & \sum_{\phi, \tilde{\chi}} \beta_{t \mid \theta}^{1}(\tilde{\chi}, \phi)\left[\hat{P}_{t}^{1}(\theta, \phi)+\left[\hat{\chi}_{t}^{1}(\theta, \tilde{\chi}, \phi)-\hat{x}_{t}(\theta)\right]\left[\hat{x}_{t}^{1}(\theta, \tilde{\chi}, \phi)-\hat{x}_{t}(\theta)\right]^{T}\right]+  \tag{30}\\
& +\sum_{\phi, \tilde{\chi}} \beta_{t \mid \theta}^{0}(\tilde{\chi}, \phi)\left[\hat{P}_{t}^{0}(\theta, \phi)+\left[\hat{x}_{t}^{0}(\theta, \tilde{\chi}, \phi)-\hat{x}_{t}(\theta)\right]\left[\hat{x}_{t}^{0}(\theta, \tilde{\chi}, \phi)-\hat{x}_{t}(\theta)\right]^{T}\right]
\end{align*}
$$

with:

$$
\begin{align*}
& \beta_{t \mid \theta}^{\kappa}(\phi, \tilde{\chi})=\beta_{t}^{\kappa}(\phi, \tilde{\chi}, \theta) / p_{\theta_{t} \mid Y_{t}}(\theta)  \tag{31}\\
& \hat{x}_{t}^{\kappa}(\theta, \phi, \tilde{\chi})=\bar{x}_{t}^{\kappa}(\theta)+K_{t}^{\kappa}(\theta, \phi) \mu_{t}^{\kappa}(\phi, \tilde{\chi}, \theta)  \tag{32.a}\\
& \hat{P}_{t}^{\kappa}(\theta, \phi)=\bar{P}_{t}^{\kappa}(\theta)-K_{t}^{\kappa}(\theta, \phi) \underline{\Phi}(\phi) H(\theta) \bar{P}_{t}^{\kappa}(\theta) \tag{32.b}
\end{align*}
$$

where:

$$
\begin{align*}
& \mu_{t}^{\kappa}(\phi, \tilde{\chi}, \theta)=\underline{\tilde{\chi}} y_{t}-\underline{\Phi}(\phi) H(\theta) \bar{x}_{t}^{\kappa}(\theta)  \tag{33.a}\\
& K_{t}^{\kappa}(\phi, \theta)=\bar{P}_{t}^{\kappa}(\theta) H(\theta)^{T} \underline{\Phi}(\phi)^{T} Q_{t}^{\kappa}(\phi, \theta)^{-1} \text { if } \quad \phi \neq\left[\begin{array}{l}
0 \\
0
\end{array}\right], \\
& =0 \quad \text { if } \phi=\left[\begin{array}{l}
0 \\
0
\end{array}\right]  \tag{33.b}\\
& \bar{x}_{t}^{\kappa}(\theta)=\bar{x}_{t}(\theta)-\kappa K_{t}(\theta)[I \vdots-I] H(\theta) \bar{x}_{t}(\theta) \tag{34.a}
\end{align*}
$$

$$
\begin{align*}
& \bar{P}_{t}^{\kappa}(\theta)=\bar{P}_{t}(\theta)-\kappa K_{t}(\theta)[I \vdots-I] H(\theta) \bar{P}_{t}(\theta)  \tag{34.b}\\
& K_{t}(\theta)=\bar{P}_{t}(\theta) H(\theta)^{T}\left[\begin{array}{c}
I \\
-I
\end{array}\right] Q_{t}(\theta)^{-1}  \tag{34.c}\\
& Q_{t}(\theta)=[I \vdots-I]\left[H(\theta) \bar{P}_{t}(\theta) H(\theta)^{T}\right]\left[\begin{array}{c}
I \\
-I
\end{array}\right]+R(\theta)  \tag{34.d}\\
& F_{t}^{\kappa}(\phi, \tilde{\chi}, \theta)=\exp \left\{-\frac{1}{2} \mu_{t}^{\kappa}(\phi, \tilde{\chi}, \theta)^{T} Q_{t}^{\kappa}(\phi, \theta)^{-1} \mu_{t}^{\kappa}(\phi, \tilde{\chi}, \theta)\right\} /\left[(2 \pi)^{m D(\phi)} \operatorname{Det}\left\{Q_{t}^{\kappa}(\phi, \theta)\right\}\right]^{\frac{1}{2}}, \text { if } \phi \neq\{0\}^{2}  \tag{35.a}\\
& =1 \quad, \text { if } \phi=\{0\}^{2} \\
& Q_{t}^{\kappa}(\phi, \theta)=\underline{\Phi}(\phi)\left(H(\theta) \bar{P}_{t}^{\kappa}(\theta) H(\theta)^{T}+G(\theta) G(\theta)^{T}\right) \Phi(\phi)^{T}  \tag{35.b}\\
& c_{t}(\theta)=\frac{|R(\theta)|^{\frac{1}{2}}}{\left|Q_{t}(\theta)\right|^{\frac{1}{2}}} \cdot \exp \left\{-\frac{1}{2} \bar{x}_{t}(\theta)^{T} H(\theta)^{T}\left[\begin{array}{c}
I \\
-I
\end{array}\right] Q_{t}(\theta)^{-1}[I \vdots-I] H(\theta) \bar{x}_{t}(\theta)\right\} \tag{36}
\end{align*}
$$

Proof outline: Eqs. (28) through (31) follow from integrations over (22). Completing the square in equation (20) yields (34.a,b,c,d) and (36). Completing the square in equation (24) and (25) yield (32.a,b), (33.a,b) and (35.a,b).
Q.E.D.

## 6 Joint IMM Coupled PDA filter with Resolution

Proposition 4 and Theorem 2 provide conditional characterizations for the joint targets modes and states. Here we use these equations to specify the JIMMCPDAR filter algorithm (this acronym stands for Joint IMM Coupled PDA Resolution). A filter cycle starts with, for each $\theta \in\{1, \ldots, N\}^{2}$, conditional mode probability $p_{\theta_{t-1} \mid Y_{t-1}}(\theta)$ and conditional mean and covariance :

$$
\begin{aligned}
& \hat{x}_{t-1}(\theta) \stackrel{\Delta}{=} E\left\{x_{t-1} \mid \theta_{t-1}=\theta, Y_{t-1}\right\}, \text { and } \\
& \hat{P}_{t-1}(\theta) \stackrel{\Delta}{=} E\left\{\left[x_{t-1}-\hat{x}_{t-1}(\theta)\right]\left[x_{t-1}-\hat{x}_{t-1}(\theta)\right]^{T} \mid \theta_{t-1}=\theta, Y_{t-1}\right\}
\end{aligned}
$$

One filter cycle consists of the following seven steps.

## JIMMCPDAR Step 1: Interaction

Equal to step 1 of JIMMCPDA [14].

## JIMMCPDAR Step 2: Prediction

Equal to step 2 of JIMMCPDA [14].

JIMMCPDAR Step 3: Merging prediction:
For all $\theta \in\{1, \ldots, N\}^{2}$, evaluate equations (34.a,b,c,d).

## JIMMCPDAR Step 4: Gating:

Following [10], now per $\kappa$ value:
Let $\bar{Q}_{t}^{\kappa, i}(\theta)$ be the i-th $m \times m$ diagonal block matrix of the $\kappa$-conditional predicted $\bar{Q}_{t}^{\kappa}(\theta)$, with

$$
\begin{equation*}
\bar{Q}_{t}^{\kappa}(\theta)=H(\theta) \bar{P}_{t}^{\kappa}(\theta) H(\theta)^{T}+G(\theta) G(\theta)^{T} \tag{37}
\end{equation*}
$$

Identify for each target $i$ and $\kappa$-value the mode $\bar{\theta}_{t}^{\kappa, i}$ for which Det $\bar{Q}_{t}^{\kappa, i}(\theta)$ is largest:

$$
\begin{equation*}
\bar{\theta}_{t}^{\kappa, i}=\underset{\theta}{\operatorname{Argmax}}\left\{\operatorname{Det} \bar{Q}_{t}^{\kappa, i}(\theta)\right\} \tag{38}
\end{equation*}
$$

and identify for each target $i$ a $\kappa$-dependent gate $G_{t}^{\kappa, i} \in \mathrm{R}^{m}$ as follows:

$$
\begin{equation*}
G_{t}^{\kappa, i}=\left\{z^{i} \in \mathrm{R}^{m} ;\left[z^{i}-h^{i}\left(\bar{\theta}_{t}^{\kappa, i}\right) \bar{x}_{t}^{\kappa, i}\left(\bar{\theta}_{t}^{\kappa, i}\right)\right]^{T} \cdot \bar{Q}_{t}^{\kappa, i}\left(\bar{\theta}_{t}^{\kappa, i}\right)^{-1} \cdot\left[z^{i}-h^{i}\left(\bar{\theta}_{t}^{\kappa, i}\right) \bar{x}_{t}^{\kappa, i}\left(\bar{\theta}_{t}^{\kappa, i}\right)\right] \leq v\right\} \tag{39}
\end{equation*}
$$

with $v$ the gate size. If the $j$-th measurement $y_{t}^{j}$ falls outside gate $G_{t}^{\kappa, i}$; i.e. $y_{t}^{j} \notin G_{t}^{\kappa, i}$, then under $\kappa_{t}=\kappa$ the $j$-th component of the $i$-th row of $\left[\Phi(\phi)^{T} \tilde{\chi}\right]$ is assumed to equal zero at moment $t$. Per $\kappa$-value this reduces the set of possible detection/permutation hypotheses to be evaluated for $\tilde{\chi}_{t}$ given $\phi_{t}=\phi$ to $\tilde{\chi} \in \tilde{X}_{t}^{\kappa}(\phi)$.

JIMMCPDAR Step 5: Hypothesis evaluation.
Using (23) as approximation and adapting the $P_{\mathrm{d}}^{i}$ and $P_{\mathrm{d}}^{0}$ in (10) for reduced detection probability due to limited gate size $v$ yields:

$$
\begin{align*}
\beta_{t}^{\kappa}(\phi, \tilde{\chi}, \theta) & \cong F_{t}^{\kappa}(\phi, \tilde{\chi}, \theta) c_{t}(\theta)^{\kappa} \lambda^{(2-D(\phi))} \cdot\left[p_{\phi_{t} \mid k_{t}}(\phi \mid \kappa)-\kappa p_{\phi_{t} \mid \kappa_{t}}(\phi \mid 0)\right] p_{\theta_{t} \mid Y_{t-1}}(\theta) / c_{t} & \text { for } \tilde{\chi} \in \tilde{X}_{t}^{\kappa}(\phi)  \tag{40}\\
& =0 & \text { for } \tilde{\chi} \notin \tilde{X}_{t}^{\kappa}(\phi)
\end{align*}
$$

$$
\begin{aligned}
p_{\phi \mid} \mid \kappa_{t}
\end{aligned}(\phi \mid \kappa)=\left[\prod_{i=1}^{2}\left(1-P_{\mathrm{d}}^{i} \cdot \operatorname{Chi}_{m}^{2}(v)\right)^{(1-\phi)}\left(P_{\mathrm{d}}^{i} \cdot \operatorname{Chi}_{m}^{2}(v)\right)^{\phi}\right] \quad \begin{array}{ll}
\text { if } \kappa=0, \phi \in\{0,1\}^{2} \\
& =\left(1-P_{\mathrm{d}}^{0} \cdot \operatorname{Chi}_{m}^{2}(v)\right)^{(1-D(\phi))}\left(P_{\mathrm{d}}^{0} \cdot \operatorname{Chi}_{m}^{2}(v)\right)^{D(\phi)}
\end{array} \begin{array}{ll}
\text { if } \kappa=1, \phi \in\left\{\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{l}
\frac{1}{2} \\
\frac{1}{2}
\end{array}\right]\right\} \\
& =0
\end{array}
$$

with $\mu_{t}^{\kappa}(\phi, \tilde{\chi}, \theta), F_{t}^{\kappa}(\phi, \tilde{\chi}, \theta), Q_{t}^{\kappa}(\phi, \theta)$ and $c_{t}(\theta)$ to be evaluated by (33.a), (35.a,b) and (36) respectively, with $c_{t}$ normalizing $\beta_{t}^{\kappa}(\phi, \tilde{\chi}, \theta)$, and $\mathrm{Chi}^{2}(\cdot)$ the Chi-squared cumulative distribution function with $m$ degrees of freedom.

JIMMCPDAR Step 6: Measurement-based update, by evaluating equations (28) - (32) and (33.b).

JIMMCPDAR Step 7: Output equations:

$$
\begin{align*}
& \hat{x}_{t}=\sum_{\theta \in\{1, \ldots, N\}^{2}} p_{\theta_{t} \mid Y_{t}}(\theta) \cdot \hat{x}_{t}(\theta)  \tag{42}\\
& \hat{P}_{t}=\sum_{\theta \in\{1, \ldots, N\}^{2}} p_{\theta_{t} \mid Y_{t}}(\theta)\left(\hat{P}_{t}(\theta)+\left[\hat{x}_{t}(\theta)-\hat{x}_{t}\right] \cdot\left[\hat{x}_{t}(\theta)-\hat{x}_{t}\right]^{T}\right) \tag{43}
\end{align*}
$$

## 7 Track-coalescence-avoiding JIMMCPDAR filter

A shortcoming of JIMMCPDA is its sensitivity to track coalescence. With the JIMMCPDA* approach, [15] has shown that this is due to JIMMCPDA's merging over permutation hypotheses, and that a suitable hypothesis pruning may provide an effective countermeasure. In order to develop such a pruning for JIMMCPDAR we need to introduce some additional processes and notation. Following [8], $\tilde{\chi}_{t}$ can be written as:

$$
\begin{equation*}
\tilde{\chi}_{t}=\chi_{t} \Phi\left(\psi_{t}\right) \quad \text { if } \quad D_{t}>0 \tag{44}
\end{equation*}
$$

where $\chi_{t}$ is a $D_{t} \times D_{t}$ permutation matrix, which is conditionally independent of $\phi_{t}$ given $D_{t}$, and where $\psi_{t}=\left[\psi_{1, t} \ldots \psi_{L_{t}, t}\right]^{T}$, with $\psi_{i, t} \in\{0,1\}$ the target indicator at moment $t$ for measurement $i$, which assumes the value one if measurement $i$ belongs to a detected target and zero if measurement $i$ comes from clutter. $\psi_{t}$ is conditionally independent of $\phi_{t}$ and $\chi_{t}$ given $D_{t}$ and $L_{t}$. Moreover, $\left\{\chi_{t}\right\}$ and $\left\{\psi_{t}\right\}$ are i.i.d. sequences.

The JIMMCPDA* filter equations are obtained from the JIMMCPDA algorithm by pruning per ( $\phi_{t}, \psi_{t}, \theta_{t}$ ) -hypothesis all except the most likely $\chi_{t}$-hypothesis prior to measurement updating.

The physical explanation why this is working for two targets has been given by [5]. If targets move closely spaced for a longer period of time, then the pdf of the individual targets tend to become of a symmetric form invariant against a permutation of the objects. Because for two targets permutations are possible for $\kappa_{t}=0$ and $\phi_{t}=\left[\begin{array}{ll}1 & 1\end{array}\right]^{T}$ only, the JIMMCPDA* hypothesis pruning strategy needs to be extended for that case only. For $\kappa=0$ and $\phi=\left[\begin{array}{ll}1 & 1\end{array}\right]^{T}$ evaluate all ( $\psi, \theta$ ) hypotheses and prune per such ( $\psi, \theta$ ) -hypothesis all except the most likely $\chi$ hypothesis. To do so, define for every $\psi$ and $\theta$ a mapping $\hat{\chi}_{t}(\psi, \theta)$ :

$$
\begin{equation*}
\hat{\chi}_{t}(\theta, \psi) \stackrel{\Delta}{\underline{\Delta}} \underset{\chi}{\operatorname{Argmax}} \beta_{t}^{0}\left(\theta, \chi \Phi(\psi),[11]^{T}\right) \tag{45.a}
\end{equation*}
$$

with maximization over all permutation matrices $\chi$ of size $2 \times 2$. The strategy of evaluating for $\kappa=0$ and $\phi=\left[\begin{array}{ll}1 & 1\end{array}\right]^{T}$ all $(\psi, \theta)$-hypotheses and only one $\chi$-hypothesis implies that we adopt per $(\phi, \tilde{\chi}, \theta)$-hypothesis the following hypothesis weights $\hat{\beta}_{t}^{\kappa}(\theta, \tilde{\chi}, \phi)$ :

$$
\begin{array}{cll}
\hat{\beta}_{t}^{\kappa}(\theta, \chi \Phi(\psi), \phi)=\beta_{t}^{\kappa}(\theta, \chi \Phi(\psi), \phi) / \hat{c}_{t} & & \text { if } D(\phi) \leq 1 \text { or if } D(\phi)=2 \text { and } \chi=\hat{\chi}(\theta, \psi)  \tag{45.b}\\
=0 & & \text { else }
\end{array}
$$

with $\hat{c}_{t}$ a normalization constant for $\hat{\beta}_{t}^{\kappa}$; i.e. such that

$$
\sum_{\kappa, \phi, \bar{\chi}, \theta} \hat{\beta}_{t}^{\kappa}(\theta, \tilde{\chi}, \phi)=1
$$

Inserting these particular weights within JIMMCPDAR, yields JIMMCPDAR*, consisting of the following cycle of of 8 steps (the first five are equivalent to the first five JIMMCPDAR steps):

## JIMMCPDAR* Steps 1-5:

Equivalent to JIMMCPDAR Steps 1-5.

JIMMCPDAR* Step 6: Hypothesis pruning.
Evaluate the new hypothesis weights using eqs. (45.a,b).

JIMMCPDAR* Step 7: Measurement-based update:
Equivalent to JIMMCPDAR Step 6, but with $\beta_{t}^{\kappa}$ replaced by $\hat{\beta}_{t}^{\kappa}$.

JIMMCPDAR*Step 8: Output equations:
Equivalent to JIMMCPDAR Step 7.

## 8 Discussion of results

This paper incorporated resolution submodels of [3] and [5] within the descriptor system approach of [8], [14] and [15] towards filtering two Markov jump linear targets from possibly unresolved, missing and false observations. For this descriptor system we developed the exact Bayesian measurement update equations (Theorem 1), and novel filter algorithms, JIMMCPDAR and JIMMCPDAR*. For closely spaced targets these filter algorithms differ significantly from the JIMMCPDA and JIMMCPDA* filter algorithms. In follow-up work the effectivity of these new filters will be shown through Monte Carlo simulations relative to JIMMCPDA and JIMMCPDA*, and relative to a good particle filter. For such particle filter the exact Bayesian filter equations of Theorem 1 will be used.

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## Appendix A Acronyms

| CPDA | Coupled PDA |
| :--- | :--- |
| CPDA* | Track-coalescence-avoiding CPDA |
| IMM | Interacting Multiple Model |
| IMMJPDA | Interacting Multiple Model Joint Probabilistic Data Association |
| IMMJPDA* | Track-coalescence-avoiding IMMJPDA |
| IMMPDA | Interacting Multiple Model Probabilistic Data Association |
| JIMMCPDA | Joint Interacting Multiple Model Coupled Probabilistic Data Association |
| JIMMCPDA* | Track-coalescence-avoiding JIMMCPDA |
| JIMMCPDAR | JIMMCPDA with Resolution |
| JIMMCPDAR* | Track-coalescence-avoiding JIMMCPDAR |
| JPDA | Joint PDA |
| JPDA* | Track-coalescence-avoiding JPDA |
| MHT | Multiple Hypotheses Tracking |
| PDA | Probabilistic Data Association |

## Appendix B List of Symbols

$a^{i}\left(\theta^{i}\right) \quad$ Target $i$ 's state transition matrix of size $n \times n$ as a function of mode $\theta^{i}$
$A(\theta) \quad$ Joint targets state transition matrix as a function of joint mode $\theta$
$b^{i}\left(\theta^{i}\right) \quad$ Target $i^{\prime}$ s state noise gain matrix of size $n \times n^{\prime}$ as a function of mode $\theta^{i}$
$B(\theta)$ Joint targets state noise gain matrix as a function of joint mode $\theta$
$D_{t} \quad$ Total number of detected targets at moment $t$
$F_{t} \quad$ Total number of false measurements at moment $t$
$\phi_{i, t} \quad$ Detection indicator for target $i$ at moment $t$
$\phi_{t} \quad$ Detection indicator vector at moment $t$, containing the detection indicators for all targets at moment $t$
$\Phi \quad$ Matrix operator to link the detection indicator vector with the measurement model
$g^{i}\left(\theta^{i}\right) \quad$ Target $i$ 's measurement noise gain matrix of size $m \times m^{\prime}$ as a function of mode $\theta^{i}$
$G(\theta)$ Joint targets measurement noise gain matrix as a function of joint mode $\theta$
$\tilde{\chi}_{t} \quad(0,1)$-matrix that is used to randomly select target measurements from the measurement vector $y_{t}$
$\underline{\chi}_{t} \quad$ "Inflated" $\tilde{\chi}_{t}$ matrix of proper size such that it randomly selects target measurements from the measurement vector $y_{t}$ by means of matrix multiplication.
$h^{i}\left(\theta^{i}\right) \quad$ Target $i$ 's state-to-measurement transition matrix of size $m \times n$ as a function of mode $\theta^{i}$
$H(\theta)$ Joint targets state-to-measurement transition matrix as a function of joint mode $\theta$
$I \quad$ Unit-matrix
$I_{r} \quad$ Diagonal matrix with its $i$-th diagonal equal to the $i$-th element of the vector $r$
$\kappa_{t} \quad$ Merging indicator at moment $t$
$L_{t} \quad$ The number of measurements at moment $t$
$\lambda \quad$ Spatial density of false measurements
$M \quad$ Total number of targets
M Set of possible modes of target
$N \quad$ Total number of modes of a target
$P_{d}^{i} \quad$ Detection probability of target $i$
$\Pi_{\eta \theta} \quad$ Transition probability of a target switching from mode $\eta$ to mode $\theta$
$\Pi \quad$ Transition probability matrix
$\psi_{i . t} \quad$ Target indicator for measurement $i$ at moment $t$
$\psi_{t} \quad$ Target indicator vector at moment $t$, containing the target indicators for all measurements at moment $t$
$r_{i} \quad$ Resolution capability factor for the $i$-th noise component of a potential target measurement
$R(\theta) \quad$ Resolution capability matrix of size $m \times m$
$\theta_{t}^{i} \quad$ Mode of target $i$ at moment $t$
$\theta_{t} \quad$ Joint targets mode at moment $t$
$v_{t}^{i} \quad$ Sequence of i.i.d. standard Gaussian variables of dimension $m^{\prime}$ representing the measurement noise for target $i$
$v_{t} \quad$ Joint targets measurement noise vector
$v_{t}^{*} \quad$ Column vector of $F_{t}$ i.i.d. false measurements
$V \quad$ Volume of the observed region
$w_{t}^{i} \quad$ Sequence of i.i.d. standard Gaussian variables of dimension $n^{\prime}$ representing the system noise for target $i$
$w_{t} \quad$ Joint targets system noise vector
$x_{t}^{i} \quad n$-vectorial state of target $i$ at moment $t$
$x_{t} \quad$ Joint targets state vector at moment $t$
$y_{t}^{k} \quad k$-th measurement at moment $t$
$y_{t} \quad$ Measurement vector at moment $t$, containing all measurements at moment $t$
$Y_{t} \quad \sigma$-algebra generated by measurements up to and including moment $t$
$z_{t}^{i} \quad m$-vectorial potential measurement of target $i$ at moment $t$
$z_{t} \quad$ Joint measurements vector at moment t , containing the potential measurements of all targets at moment $t$
$\tilde{z}_{t} \quad$ Joint measurements vector at moment t , containing the potential measurements of all detected targets at moment $t$ in a fixed order


[^0]:    ${ }^{1}$ In [17] a version of JIMMCPDA for independently maneuvering targets is developed under the name IMMJPDA-Coupled filter.

