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Nu-Gap Metric A Sum-Of-Squares and Linear Matrix Inequality Approach

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Executive summary

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Problem area

The nu-gap metric represents a good measure of the distance between systems in a closed-loop setting. The purpose of this paper is to present a novel method to compute the nu-gap using a Semi-Definite Programming (SDP) procedure.

Description of work

Our approach is formulated through a three-step modus operandi: (i) first an initial central transfer function is computed through Linear Matrix Inequality (LMI) relaxations of a non-convex problem, on the basis of matrix Sum-Of-Squares (SOS) decompositions, followed by (ii) a non-linear LMI-based refinement, and finally (iii) the actual computation of the nu-gap using the Kalman-Yakubovich-Popov (KYP) Lemma. We illustrate the practicality of the proposed method on numerical examples.

Results and conclusions

The main benefits of our approach are as follows. First, it is well known that the nu-gap metric does not account for any performance objectives of a closed-loop system. Hence, if the application at hand includes also robust performance specifications, and since our optimization problem is SDP based, one could easily add additional performance constraints to the nu-gap by ensuring that the appropriate sensitivity functions are all well behaved. Second, our approach could potentially be combined with other LMI based applications in the nu-gap metric, such as the recent and powerful results related to model order reduction.

Applicability

System modeling and design of (flight) control systems.

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A Sum-Of-Squares and Linear Matrix Inequality Approach

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Abstract—The nu-gap metric represents a good measure of the distance between systems in a closed-loop setting. The purpose of this paper is to present a novel method to compute the nu-gap using a Semi-Definite Programming (SDP) procedure. Our approach is formulated through a three-step modus operandi: (i) first an initial central transfer function is computed through Linear Matrix Inequality (LMI) relaxations of a nonconvex problem, on the basis of matrix Sum-Of-Squares (SOS) decompositions, followed by (ii) a non-linear LMI-based refinement, and finally (iii) the actual computation of the nu-gap using the Kalman-Yakubovich-Popov (KYP) Lemma. We illustrate the practicality of the proposed method on numerical examples.

I. Introduction

Gap and graph metrics [1] have been known to provide a measure of the separation between open-loop systems, in terms of their closed-loop behavior. The first attempt to introduce such a metric, simply known as gap metric, was formulated in [2], [3], whereas an efficient method for computing it was presented in [4], with recent works from a fairly general perspective proposed in [5]. Other significant metrics have also been investigated, such as (i) the T-gap metric [6], (ii) the pointwise gap [7], and (iii) Vinnicombe's popular nu-gap (i.e. ν-gap) metric [8], [9]. Similar to its predecessor gap metrics, the ν-gap provides also a means of quantifying feedback system stability and robustness, while being concurrently less conservative and simpler to compute. Time-varying and nonlinear extensions to both the gap metric [10], [11], [12], [13] and the ν-gap metric [14], [15], [16] have also been researched, although analytical computations of these metrics, in this nonlinear setting, is generally difficult. Over the years the use of these metrics has received much attention. In particular, the ν-gap was extensively studied in the realm of system identification [17], [18], [19], model order reduction [20], [21], [22], [23], [24], [25], and robust control [26], [9].

The purpose of this paper is to present a novel method to compute Vinnicombe's ν-gap. Although being sub-optimal, the presented method consists in expressing the ν-gap computation as different Linear Matrix Inequality (LMI) [27] optimizations sub-problems - subsequently formulated as Semi-Definite Program (SDP) [28] - for which there are several powerful numerical solutions [29], [30].

The main benefits of our approach are as follows. First, it is well known that the ν-gap metric does not account for any performance objectives of a closed-loop system. Hence, if the application at hand includes also robust performance specifications, and since our optimization problem is SDP based, one could easily add additional performance constraints by ensuring that the appropriate sensitivity functions are all well behaved, see for example [23]. Second, our approach could potentially be combined with other LMI based applications in the ν-gap metric, such as the recent and powerful results related to model order reduction [23], [24], [25]. Our algorithm is based on a three-step modus operandi: (i) an initial central transfer function is first computed through LMI relaxations of a nonconvex problem, on the basis of matrix Sum-Of-Squares (SOS) decompositions, followed by (ii) a non-linear SDP based refinement, and finally (iii) the actual computation of the ν-gap using the Kalman-Yakubovich-Popov (KYP) Lemma.

The remainder of the paper is organized as follows. In Section II, both (i) the KYP Lemma, and (ii) SOS definitions are first recalled. In Section III, the computation of the ν-gap, through an LMI approach, is defined. In Section IV, An LMI based solution, through the use of the KYP Lemma, is presented. In Section V, the central transfer function is computed. In Section VI, numerical experiments are evaluated. Finally, conclusions and future directions are presented in Section VII.

The nomenclature is fairly standard. M^T and M^* denote the transpose and complex-conjugate transpose of a real or complex matrix M. Matrix inequalities are considered in the sense of *Löwner*. Further $\lambda(M)$ denotes the zeros of the characteristic polynomial $\det(sI - M) = 0$. $\bar{\lambda}(M)$ is the maximum eigenvalue of M. $\mathcal{L}_\infty$ is the *Lebesgue* normed space s.t. $\|G\|_\infty := ess \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)) < \infty$, with $\bar{\sigma}(G)$ the largest singular value of matrix $G(\cdot)$. Similarly, $\mathcal{H}_\infty \subset \mathcal{L}_\infty$ is the *Hardy* normed space s.t. $\|G\|_\infty := \sup_{Re(s)>0} \bar{\sigma}(G(s))$. $\mathcal{RL}_\infty$ (resp. $\mathcal{RH}_\infty$) represent the subspace of real rational Transfer

Functions (TFs) in $\mathcal{L}_\infty$ (resp. $\mathcal{H}_\infty$). Finally I and 0 will be used to denote the identity and null matrices respectively, assuming appropriate sizes.

II. PRELIMINARIES

This section introduces first the KYP Lemma [31] (see also [32] for a proof) and the Bounded Real Lemma (BRL) [33].

Lemma 1: Let complex matrices A, B, and a symmetric matrix Θ, of appropriate sizes, be given. Suppose $\lambda(A) \subset \mathbb{C}^- \cup \mathbb{C}^+$, then the following two statements are equivalent.

(i)

$$\begin{bmatrix} (j\omega - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega - A)^{-1}B \\ I \end{bmatrix} < 0 \qquad (1)$$

(ii) There exists a matrix $P = P^*$, for which the LMI for the following linear matrix map $L(P)$ holds.

$$L(P) + \Theta < 0$$
$$L(P) = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \qquad (2)$$

Proof: See [32].

Remark 1: We have dealt here with the strict version of the KYP lemma, i.e. strict inequalities, since no controllability/stabilizability assumptions become necessary, and with use of interior-point based solvers, existence of strictly feasible solutions will be guaranteed.

Remark 2: If matrices A, B, and Θ are all real, the equivalence still holds when restricting P to be real [34].

Lemma 2: Let a transfer function $G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ be given, then the following statements are equivalent.

(i) $\forall \gamma > 0, \ \lambda(A) \subset \mathbb{C}^- \cup \mathbb{C}^+, \ \|G\|^2 < \gamma^2$ (3)

(ii) $\exists P \mid P = P^*$ and $L(P) + \Theta < 0$

$$L(P) = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$
$$\Theta = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \qquad (4)$$

Proof: Using Lemma 1. This result is also known as the BRL in LMI form [35]. Note that for the case where $\lambda(A) \subset \mathbb{C}^-$ we need to add the stability constraint $P > 0$ in (4), and for the case where $\lambda(A) \subset \mathbb{C}^0$ it is standard practice to perturb A by $-\epsilon I$, with $0 < \epsilon \ll 1$.

Next we briefly recall here some SOS related results. A SOS is a convex, finite dimensional, optimization problem, with a linear cost and SOS constraints [36], [37]. A scalar multivariate polynomial $p(x)$, with $x \in \mathbb{R}^n$, may be written as $p(x) := \sum_{i=1}^N c_i x_1^{\alpha_{i,1}} x_2^{\alpha_{i,2}} \cdots x_n^{\alpha_{i,n}}$, with $c_i \in \mathbb{R}$ the coefficients, and $\alpha_{i,j} \in \mathbb{N}_0 := (0 \cup \mathbb{N})$, $i = 1 \cdots N, j = 1 \cdots n$ the exponents. The set of all polynomials in the variable x is denoted by $\mathbb{R}[x]$. Polynomials are often expressed in terms of a finite linear

combination of their monomials (products of powers of x_j), these latter denoted by $k(x) = x_1^{\alpha_{i,1}} x_2^{\alpha_{i,2}} \cdots x_n^{\alpha_{i,n}}$. For $\alpha \in \mathbb{R}^n$ it is often standard practice to instead use the following notation $k(x) := x^\alpha$. Recent extensions to so-called *matrix* SOS decompositions have been derived and are restated hereafter.

Definition 1: A symmetric matrix-valued polynomial matrix $S(x) : \mathbb{R}^n \to \mathcal{S}^r$ is said to be SOS, if there exists a (not necessarily square and typically tall) polynomial matrix $T(x)$ such that $S(x) = T(x)^T T(x)$ [38], [39], [40].

III. PROBLEM STATEMENT

There exists several equivalent definitions of the ν-gap metric. The one chosen in this paper is most convenient for our purpose. The ν-gap between two plants having transfer functions P_1 and P_2, with $P_1, P_2 \in \mathcal{R}$, is defined as [8]

$$\delta_\nu(P_1, P_2) := ...$$
$$\begin{cases} \delta_{\mathcal{L}_2}(P_1, P_2) & \text{if the WNC holds} \\ 1. & \text{else} \end{cases} \qquad (5)$$

with

$$\delta_{\mathcal{L}_2}(P_1, P_2) := \|(I + P_2 P_2^*)^{-1/2}(P_2 - P_1)(I + P_1^* P_1)^{-1/2}\|_\infty \quad (6)$$

and *WNC* the so-called *Winding Number Condition* associated with the Nyquist diagram, for which an efficient computational method already exists [8], [9]. This paper focuses on the $\delta_{\mathcal{L}_2}(P_1, P_2)$ part of the ν-gap metric, also expressed as

$$\delta_{\mathcal{L}_2}(P_1, P_2) = ...$$

$$\sup_{\omega \in \mathbb{R}} \left[\bar{\lambda}\left((I + P_1^* P_1)^{-1}(P_2 - P_1)^*(I + P_2 P_2^*)^{-1}(P_2 - P_1) \right) \right]^{(1/2)} \quad (7)$$

Our goal consists now in computing (7) using a SDP approach, albeit through a sub-optimal approach (the sub-optimality aspects of our approach will further be discussed in Section V). Now (7) can easily be recast into LMI form, as the minimization of a maximum eigenvalue problem [27] (or $\mathcal{L}_2$-induced gain of a static operator).

Problem 1 Let transfer functions $P_1, P_2 \in \mathcal{R}$ be given, then the solution to (7) is also given by

$$\delta_{\mathcal{L}_2}^2(P_1, P_2) := \inf_{\omega \in \mathbb{R} \ \lambda \in \mathbb{R}^+} \lambda \quad \text{subject to ...}$$
$$(P_2 - P_1)^*(I + P_2 P_2^*)^{-1}(P_2 - P_1) < \lambda.(I + P_1^* P_1)$$

From Problem 1 we obtain the following result.

Lemma 3: Let P_1, P_2 be given. Let a so-called central transfer function T be such that $T^* T := I + P_2 P_2^*$, and a transfer function L be such that $L := T^{-1}$, with $P_1, P_2, T, L \in \mathcal{R}$, then the solution $\delta_{\mathcal{L}_2}(P_1, P_2)$ to Problem 1 is given by

$$\delta_{\mathcal{L}_2}^2(P_1, P_2) := \inf_{\omega \in \mathbb{R} \ \lambda \in \mathbb{R}^+} \lambda \quad \text{subject to ...}$$
$$\begin{bmatrix} L(P_2 - P_1) \\ P_1 \\ I \end{bmatrix}^* \begin{bmatrix} I & 0 & 0 \\ 0 & -\lambda.I & 0 \\ 0 & 0 & -\lambda.I \end{bmatrix} \begin{bmatrix} L(P_2 - P_1) \\ P_1 \\ I \end{bmatrix} < 0 \qquad (8)$$

Proof: Define $P_2 := \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ and $T := \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right]$. Since by definition $I + P_2 P_2^* > 0$, $\exists T \mid T^* T = I + P_2 P_2^*$. Again since $D_T = I + D_2 D_2^* > 0$, we infer D_T^{-1} to be well-defined, thus we can find a realization for L, such that $L = T^{-1}$ [41]. Now by rewriting Problem 1 as partitioned matrices we get (8).

IV. APPLICATION OF THE KYP LEMMA

Now we express (8) in a form which is amenable to the KYP paradigm.

Lemma 4: Let $P_1 := \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$, $P_2 := \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ be given. Let the so-called central transfer function be defined as $T := \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right]$ such that $T^* T = I + P_2 P_2^*$, and define the following additional transfer functions as $L := \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right]$

such that $L := T^{-1} = \left[\begin{array}{c|c} A_T - B_T D_T^{-1} C_T & B_T D_T^{-1} \\ \hline -D_T^{-1} C_T & D_T^{-1} \end{array}\right]$,

$R := \left[\begin{array}{c|c} A_R & B_R \\ \hline C_R & D_R \end{array}\right]$ such that $R := L(P_2 - P_1) =$

$\left[\begin{array}{ccc|c} A_L & B_L C_2 & -B_L C_1 & B_L(D_2 - D_1) \\ 0 & A_2 & 0 & B_2 \\ 0 & 0 & A_1 & B_1 \\ \hline C_L & D_L C_2 & -D_L C_1 & D_L(D_2 - D_1) \end{array}\right]$, and $S :=$

$\left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right]$ such that $S := \left[\begin{array}{c} R \\ P_1 \end{array}\right] = \left[\begin{array}{cc|cc} A_R & 0 & B_R & 0 \\ 0 & A_1 & 0 & B_1 \\ \hline C_R & 0 & D_R & 0 \\ 0 & C_1 & 0 & D_1 \end{array}\right]$,

with $P_1, P_2, T, L, R, S \in \mathcal{R}$, then the solution $\delta_{\mathcal{L}_2}(P_1, P_2)$ to Problem 1 is given by

$$\delta_{\mathcal{L}_2}^2(P_1, P_2) := \inf_{\substack{\omega \in \mathbb{R} \\ \lambda \in \mathbb{R}^+}} \lambda \quad \text{subject to ...}$$
$$\left[\begin{array}{c} (sI - A_S)^{-1} B_S \\ I \end{array}\right]^* \Theta \left[\begin{array}{c} (sI - A_S)^{-1} B_S \\ I \end{array}\right] < 0 \quad s = jw \quad (9)$$
$$\Theta = \left[\begin{array}{cc} C_S & D_S \\ 0 & I \end{array}\right]^* \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & -\lambda.I & 0 \\ 0 & 0 & -\lambda.I \end{array}\right] \left[\begin{array}{cc} C_S & D_S \\ 0 & I \end{array}\right]$$

Proof: The proof is straightforward. From (8), expand P_1 as $C_1(sI - A_1)^{-1} B_1 + D_1$, and similarly for P_2 and L, and regroup terms.

Now we give the following main result.

Lemma 5: Let transfer functions $P_1, P_2 \in \mathcal{R}$ be given. Let transfer function S be defined as in Lemma 4, then the following three statements are equivalent.

(i) $\delta_{\mathcal{L}_2}^2(P_1, P_2) := \| (I + P_2 P_2^*)^{-1/2} (P_2 - P_1)(I + P_1^* P_1)^{-1/2} \|_\infty^2$

(ii) $\delta_{\mathcal{L}_2}^2(P_1, P_2) := \inf_{\substack{\omega \in \mathbb{R} \\ \lambda \in \mathbb{R}^+}} \lambda \quad \text{subject to ...}$
$(P_2 - P_1)^* (I + P_2 P_2^*)^{-1} (P_2 - P_1) < \lambda.(I + P_1^* P_1)$

(iii) $\delta_{\mathcal{L}_2}^2(P_1, P_2) := \inf_{\substack{\omega \in \mathbb{R} \\ \lambda \in \mathbb{R}^+}} \lambda \quad \text{subject to ...}$
$\exists P, \ P = P^T, \ 0 < \lambda < 1, \ L(P) + \Theta < 0$

$$L(P) = \left[\begin{array}{cc} A_S & B_S \\ I & 0 \end{array}\right]^* \left[\begin{array}{cc} 0 & P \\ P & 0 \end{array}\right] \left[\begin{array}{cc} A_S & B_S \\ I & 0 \end{array}\right]$$
$$\Theta = \left[\begin{array}{cc} C_S & D_S \\ 0 & I \end{array}\right]^* \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & -\lambda.I & 0 \\ 0 & 0 & -\lambda.I \end{array}\right] \left[\begin{array}{cc} C_S & D_S \\ 0 & I \end{array}\right]$$
(10)

Proof: From (9), it is a straightforward application of Lemma 1 and Lemma 2.

Note that (iii) in Lemma 5 gives the exact solution, in LMI form, to Problem 1 and hence to (7). So far we have assumed that the central transfer function T, such that $T^* T := I + P_2 P_2^*$, could exactly be computed. To the best of our knowledge, no method is known that could deliver this exact T, in the general case. Hence, we present in the sequel a method which provides an approximation to T. It is this approximation in T that renders our computation of the ν-gap metric sub-optimal.

V. COMPUTING THE CENTRAL TRANSFER FUNCTION T

Our goal is to find the central transfer function $T \in \mathcal{R}$ such that $T^* T \approx I + P_2 P_2^*$, with the ensuing realizations of T and P_2 as defined in Lemma 3. Here too we will solve our problem in the $\mathcal{L}_\infty$ norm paradigm.

Problem 2 Let $P_2 := \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \in \mathcal{R}$ be given, with $A_2 \in \mathbb{R}^{n_x \times n_x}$, $B_2 \in \mathbb{R}^{n_x \times n_u}$, $C_2 \in \mathbb{R}^{n_y \times n_x}$, and $D_2 \in \mathbb{R}^{n_y \times n_u}$, find the optimal $\hat{T} \in \mathcal{R}$ such that

$$\hat{T} := \arg\inf_{(A_T \in \mathbb{R}^{n_x \times n_x}, B_T \in \mathbb{R}^{n_x \times n_u}, C_T \in \mathbb{R}^{n_Y \times n_x}, D_T \in \mathbb{R}^{n_Y \times n_u})} \| (I + P_2 P_2^*) - T^* T \|_\infty$$

Next, we present a solution to Problem 2.

Lemma 6: Let $P_2 := \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \in \mathcal{R}$ be given, and let transfer function T be defined as $T := \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right]$, and transfer function H be defined as $H := \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right]$ such that $H := (I + P_2 P_2^*) - T^* T =$

$\left[\begin{array}{cccc|c} -A_2^* & -C_2^* C_2 & 0 & 0 & -C_2^* D_2 \\ 0 & A_2 & 0 & 0 & B_2 \\ 0 & 0 & -A_T^* & -C_T^* C_T & -C_T^* D_T \\ 0 & 0 & 0 & A_T & B_T \\ \hline B_2^* & D_2^* C_2 & -B_T^* & -D_T^* C_T & I + D_2^* D_2 - D_T^* D_T \end{array}\right]$, then

the solution $\hat{T}$ to Problem 2 is given by

$$\hat{T} := \arg\inf f(x) \quad \text{with} \quad f(x) = \gamma$$
$$x := (\gamma, K, A_T, B_T, C_T, D_T) \quad \text{with} \quad K = K^*$$
$$\text{subject to} \quad f(x) > 0 \quad G(x) < 0$$
$$G(x) = \left[\begin{array}{ccc} A_H^* K + K A_H & K B_H & C_H^* \\ B_H^* K & -\gamma.I & D_H^* \\ C_H & D_H & -\gamma.I \end{array}\right]$$

(11)

Proof: Direct application of the BRL in LMI form to Problem 2.

Here we clearly see that the problem defined by (11) is non-convex, on the one hand due to such products as $C_T^* C_T$, $C_T^* D_T$, and on the other, due to the various cross-products between the Lyapunov matrix K and the state-space matrices of T. Hence to solve Problem 2, we aim at constructing meaningful convex relaxations. Solving Problem 2 will be done in two stages: (i) first find an accurate initial value for $\hat{T}$, and then (ii) through an iterative, nonlinear, yet SDP approach, refine this earlier guess. We present next the first stage of this method.

A. LMI relaxations on the basis of matrix SOS decompositions

In this section, we compute an initial value for $\hat{T}$ in (11). The method we use here is based upon SOS decompositions, since it is well known that a plethora of relevant non-convex optimization problems may have natural formulations (or relaxations) as SOS programs. From our presentation of SOS problems in Section II, and from Definition 1, we recognize that (11) may be viewed as a polynomial SDP, having a linear objective f(x), with coefficients c_i as symmetric matrices, and variable x defined as $x := (\gamma, K, A_T, B_T, C_T, D_T)$. Hence, we are here concerned with a *matrix* SOS decompositions [38], [39], [40]. To convert our non-convex problem in (11) to an amenable SOS representation, we will heavily rely on the work done in [39], although other approaches also exist [38], [42], [43], [44].

Remark 3: Since we have $I + D_2^* D_2 > 0$, we can always find D_T such that $D_T^* D_T = I + D_2^* D_2$ by cholesky factorization. Hence, in our case D_T is not a free variable, which simplifies (11), by reducing x to $x := (\gamma, K, A_T, B_T, C_T)$.

Lemma 7: Let $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, and $G : \mathbb{R}^n \to \mathbb{S}^r$ be general scalar, and symmetric-matrix-valued polynomials. Consider now the following problem

$$\text{minimize } f(x) \quad \text{subject to } G(x) \preceq 0 \tag{12}$$

then an LMI relaxation of (12), and hence a sub-optimal solution, based on matrix SOS decompositions, is given by the following optimization problem

$$\begin{aligned} &\text{minimize } d \quad \text{subject to ...} \\ &S(x) \succeq 0 \text{ and } s_0(x) := f(x) + \langle S(x), G(x) \rangle - d \\ &S(x) \text{ and } s_0(x) \text{ are } SOS \end{aligned} \tag{13}$$

with $S(x)$ Lagrange multiplier matrices which are polynomial functions of x.

Proof: From polynomial Lagrange duality with SOS, see p. 61 [39].

Next we recall the following important results, from [39], that will prove very useful in finding an expression for $S(x)$.

Proposition 1: If we choose pairwise different monomials $u_j(x)$, $j = 1 \cdots \zeta$, and collect them into the vector $u(x) = (u_1(x)^T \cdots u_\zeta(x)^T)^T$, then $S(x)$ is said to be SOS with respect

to the monomial basis $u(x)$, if there exists real matrices T_j, $j = 1 \cdots \zeta$, with r columns such that $S(x) = T(x)^T T(x)$ with $T(x) = \sum_{i=1}^\zeta T_j u_j(x)$. Further if $w_j(x)$, $j = 1 \cdots \eta$ denote the pairwise different monomials that appear in $u(x)u(x)^T$, then we can determine the unique symmetric matrices Z_j s.t. $u(x)u(x)^T = \sum_{j=1}^\eta Z_j w_j(x)$ [39].

Lemma 8: The polynomial matrix $S(x)$ of dimension r is SOS with respect to the monomial basis $u(x)$ if and only if there exists symmetric S_j with $S(x) = \sum_{j=1}^\eta S_j w_j(x)$, and the following linear system has a solution W

$$\langle W, I_r \otimes Z_j \rangle_r = S_j, \quad j = 1 \cdots \eta \qquad W \succeq 0 \tag{14}$$

with the matrix-valued generalization of the inner product defined by $\langle A, B \rangle_r = Trace_r(A^T B)$

$$\text{such that} \quad Trace_r(C) = \begin{bmatrix} Trace(C_{11}) & \cdots & Trace(C_{1r}) \\ \vdots & & \vdots \\ Trace(C_{r1}) & \cdots & Trace(C_{rr}) \end{bmatrix} \tag{15}$$

Proof: Using Proposition 1, see pp. 66-67 in [39].

We are now in a position to present the LMI relaxations procedure, from [39], on the basis of matrix SOS decompositions, which will allow us to compute a solution to (13), hence an approximative solution to (11), and thus a solution to Problem 2.

Proposition 2: Let $S(x)$ and $s_0(x) := f(x) + \langle S(x), G(x) \rangle - d$ be given as in Lemma 7, then

(i) Select the monomial vector $v(x) \in \mathbb{R}^\zeta$, and some real coefficient matrix $B \in \mathbb{R}^{r \times \zeta}$, such that $G(x) = B(I_r \otimes v(x))$

(ii) Choose monomial vectors $u(x)$ and $u_0(x)$ of length ζ and ζ_0 to parameterize $S(x)$ and $s_0(x)$ respectively

(iii) Find the pairwise different monomials $w_i(x)$ such that: $w_0(x) = 1$, $u_0(x)u_0(x)^T = \sum_{j=0}^\eta P_j^0 w_j(x)$, $v(x) \otimes u(x)u(x)^T = \sum_{j=0}^\eta P_j w_j(x)$, and $f(x) = \sum_{j=0}^\eta a_j w_j(x)$

(iv) Find a solution to the following LMI
$W_0 \succeq 0$, $W \succeq 0$ and
$a_j + \langle W_0, P_j^0 \rangle + \langle W, (B \otimes I_\zeta)(I_r \otimes P_j) \rangle - \delta_j d = 0$, $j = 0, 1, ..., \eta$
with $\delta_0 = 1$ and $\delta_j = 0$ for $j \geq 1$

Proof: Using Proposition 1, Lemma 8, and the derivation on pp. 71-72 [39].

Finally $u_0(x)$ ought to be chosen so that its first $n + 1$ elements are such that

$$(I_{n+1} \ 0)u_0(x) = (1 \ x^T)^T \tag{16}$$

then the optimal solution to (13) can be reconstructed from the kernel of W_0. Ideally this kernel should have dimension one, so that only a single vector is recovered, i.e. the optimal solution. If this is not the case, then the optimal solution can be found by solving a polynomial semi-definite feasibility

problem on the basis vectors of this kernel (see [39] for further details).

Remark 4: The relaxations in Lemma 7, as stated by Theorem 4.9 p. 72 in [39], are "*guaranteed to converge to the optimal value, if we choose (...) all monomials up to a certain degree, and if we let the degree bound grow to infinity*". Hence, the non-linear SDP refinement of Section V-B is, from a conceptual viewpoint, not required. However, dimensionality problems exist with SOS formulations, since the number of decision variables increases exponentially with the number of variables and the degree of the polynomials. Hence, for system's order higher than say three or four, a minimal length for $u_0(x)$ could be chosen as $u_0(x) := [1 \ \gamma \ svec(K)^T \ vec(A_T)^T \ vec(B_T)^T \ vec(C_T)^T]$, with $vec(\cdot)$ and $svec(\cdot)$ the vectorization of a matrix, and a symmetric matrix, respectively.

B. SDP-based nonlinear refinement

In Section V-A we computed an initial value for $\hat{T}$ in (11). Now, to keep the SOS decomposition computationally tractable, we had to limit the number, and degree, of monomials used in Proposition 2. To compensate for this loss of accuracy, we introduce next a computationally fast nonlinear SDP based refinement. First we recall a result, that will be very helpful to solve LMIs with quadratic equality constraints, related to the solution of the Continuous-Time Algebraic Riccati Equation (CARE) [31] as presented in [45].

Lemma 9: Let matrix $Q > 0$ be given, then the solution $\hat{X}$, of appropriate size, to the equation $Q - X^T X = 0$ is given by

$$\hat{X} := \quad \arg\min \quad -\text{trace}(X)$$
$$\text{subject to} \quad \begin{bmatrix} Q & X^T \\ X & I \end{bmatrix} \geq 0 \quad (17)$$

Proof: See [45].

We will use LMI (17) to linearize the quadratic constraint $C_T^* C_T$ in (11), by replacing $C_T^* C_T$ by Q, and then replacing X in (17) by C_T (assuming matrix C_T square or fat, which is always the case in practice), and finally by adding LMI (17) to LMI (11). Now we discuss the algorithm for our recursive procedure. Our proposed approach is basically a simple two-step iterative LMI search, with a bisection on γ, in spirit reminiscent of μ *DK*-iteration synthesis [41]. The algorithm is structured as follows

(i) Compute D_T by cholesky factorization, see Remark 3
(ii) Compute an initial value for A_T, B_T, C_T from Section V-A
(iii) In LMI (11), fix A_T, B_T, C_T, minimize γ wrt K
  (a) Set $\bar{\gamma} = \gamma$, $\underline{\gamma} = 0$, and $\gamma = \underline{\gamma} + (\bar{\gamma} - \underline{\gamma})/2$
  (b) In LMI (11) replace $C_T^* C_T$ by $Q > 0$, and in LMI (17) replace X by C_T
    minimize $-trace(C_T)$ wrt A_T, B_T, C_T, Q
    subject to LMI (11) and LMI (17)
  (c) If optimization is feasible set $\bar{\gamma} = \gamma$, otherwise $\underline{\gamma} = \gamma$

(d) Repeat from (a) until convergence
(iv) Repeat from (iii) until convergence or maximum iteration reached

Note that this method is only a heuristic for which convergence towards a global optimum, or even a local optimum, is not guaranteed. This said, in practice convergence has been achieved within 40 to 60 iterations.

VI. NUMERICAL EXPERIMENT

We provide first a brief summary of the complete method. Computation of the $\delta_{\mathcal{L}_2}(\cdot, \cdot)$ part of the nu-gap, through an SDP approach, is done as follows

(1) Compute first the central transfer function T by
  (a) Computing D_T by cholesky factorization, see Remark 3
  (b) Computing an initial value for A_T, B_T, C_T from Section V-A
  (c) Computing refined values for A_T, B_T, C_T from Section V-B
(2) Then use the obtained A_T, B_T, C_T, D_T to solve the LMI (iii) in Lemma 5 which gives the solution to Problem 1, and hence to (7)

We illustrate now our approach with numerical examples, in which all LMI problems are solved in a MATLAB® environment using YALMIP [46] together with the SeDuMi

TABLE I.   STATE-SPACE DATA FOR P_1 AND P_2

	State-Space Data			
	A	B	C	D
P_1	-1/2	1	2	-3/2
P_2 (Example 1)	1	3	1	0
P_2 (Example 2)	-1.25	4	1/2	-0.75
P_2 (Example 3)	-0.75	1.5	1.75	-1.25
P_2 (Example 4)	0.5	1	-1	6

TABLE II.   STATE-SPACE DATA FOR T (EXAMPLE 1)

	State Space Data				
	A	B	C	D	$\|(I + P_2 P_2^*) - T^* T\|_\infty$
T (SOS)	1.125	-1.114	1.135	1	5.486
T (NL OPTIM)	1.026	-1.537	1.435	1	0.081

TABLE III.   STATE-SPACE DATA FOR T (EXAMPLE 2)

	State Space Data				
	A	B	C	D	$\|(I + P_2 P_2^*) - T^* T\|_\infty$
T (SOS)	-0.954	-0.936	0.955	1.25	1.625
T (NL OPTIM)	-1.295	-0.993	3.321	1.25	0.037

TABLE IV.   STATE-SPACE DATA FOR T (EXAMPLE 3)

	State Space Data				
	A	B	C	D	$\|(I + P_2 P_2^*) - T^* T\|_\infty$
T (SOS)	0.956	-0.936	0.956	1.6	0.653
T (NL OPTIM)	0.785	-0.724	0.930	1.6	0.071

TABLE V.   STATE-SPACE DATA FOR T (EXAMPLE 4)

	State Space Data				
	A	B	C	D	$\|(I + P_2 P_2^*) - T^* T\|_\infty$
T (SOS)	1.152	1.144	1.121	6.082	40.306
T (NL OPTIM)	0.502	-0.580	1.712	6.082	0.053

TABLE VI.   COMPUTATION OF $\delta_{\mathcal{L}_2}(P_1, P_2)$

	Example 1	Example 2	Example 3	Example 4
MATLAB gapmetric	0.832	0.488	0.181	0.683
Our Method	0.832	0.493	0.180	0.684

solver [30]. The purpose is to compute $\delta_{\mathcal{L}_2}(P_1, P_2)$, for four SISO plants P_1 through P_4, and to compare it to the value obtained from the MATLAB function *gapmetric*. Model P_1 has fixed state-space data, whereas model P_2 has four sets of state-space data values corresponding to *Example 1* through *Example 4* in Table I. Although, from a conceptual viewpoint, our method can be applied to high-order systems, in this paper we only illustrate the technique on first-order systems as to keep the SOS machinery computationally tractable. Next the values of the state-space data for the central transfer function T, obtained after the SOS step of Section V-A, and after the additional nonlinear iteration of Section V-B, are given in Table II through Table V for *Example 1* through *Example 4* respectively. In addition, as an indication of the quality for the obtained transfer function T, the metric for Problem 2 given by $\|(I + P_2 P_2^*) - T^* T\|_\infty$ is also reported within these tables. For the SOS approach, we have used the following monomials in $u_0(x) := [1 \ \gamma \ svec(K)^T \ A_T \ B_T \ C_T \ A_T^2 \ B_T^2 \ C_T^2 \ A_T^4 \ B_T^4 \ C_T^4]$, and $u(x) := [1 \ svec(K)^T \ A_T \ B_T \ C_T]$. With this set of monomials relatively good starting values were obtained. Further, the nonlinear refinement exhibited excellent convergence properties, reaching low value for $\|(I + P_2 P_2^*) - T^* T\|_\infty$ in the range 0.037 to 0.081. The results for $\delta_{\mathcal{L}_2}(P_1, P_2)$ reported by the MATLAB function *gapmetric* are reported as *MATLAB gapmetric* in Table VI, whereas the method outlined in this paper is labeled as *Our Method*. As can be seen, our method provides highly accurate results.

As a final note, the optimization cost of SDPs differ depending on the tolerance level and problem size. For example for the LMI in (4), let matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, and let $n = n_x^2 + n_x$ be the number of decision variables, and m the number of rows of LMI, then the asymptotic computational complexity, or flop cost, of SDP solvers is in $O(n^2 m^{2.5} + m^{3.5})$ for SeDuMi [30], and in $O(n^3 m)$ for MATLAB LMI-lab [47]. These aspects could potentially be used to evaluate the computational cost of our method.

VII.   CONCLUSION

We have presented a novel approach to accurately compute the ν-gap metric through a Semi-Definite Programming (SDP) procedure. Since our optimization problem is SDP based, our approach may allow for the inclusion of additional constraints, such as closed-loop performance criteria. For systems of small size, the Sum-Of-Squares machinery

used here is computationally tractable. However, for larger systems, dimensionality related problems may soon occur, due to the exponential growth in decision variables.

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