Nationaal Lucht- en Ruimtevaartlaboratorium
National Aerospace Laboratory NLR

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Advanced stochastic method for probabilistic analysis

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ADVANCED STOCHASTIC METHOD FOR PROBABILISTIC ANALYSIS

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ABSTRACT

Recently at NLR a very efficient stochastic method has been implemented for probabilistic analysis. The method is a so-called Second-Order Reliability Method (SORM). The basis of the method is formed by an efficient optimization scheme applied to find the safety index (most probable point) after which second-order information is added to the solution by solving one of Tvedt's formulas. The optimization scheme is based on the approximate model concept, approximating the objective and constraints (limit-states) by means of second-order Taylor series at some starting point. Next, the approximate model can be optimized using standard optimization techniques giving a better estimate of the optimum. Around this optimum a new approximate model is formed and optimized. These steps are repeated until the optimum has been found. Special features have been applied to minimize the number of expensive limit-state function evaluations and prevent oscillation of the solution. Furthermore, an efficient scheme has been implemented to transform non-normal dependent variables to a set of normal independent variables.
The method has been implemented in a computer code called RAP (Reliability Analysis Program), which can be used on top of any existing deterministic program, e.g. finite element program or crack growth program. The accuracy, correct implementation and performance are demonstrated by two examples.

KEYWORDS

Stochastic method, probabilistic analyses, SORM, optimization.

INTRODUCTION

In this paper a method is presented to calculate the probability of a problem, of which one or more of the model parameters are random variables, i.e. statistically determined. Mathematically this can be stated as:

$$p = P(G(\underline{X}) \leq 0) = \int_{G(\underline{x})\leq 0} f(\underline{x})\, d\underline{x} \qquad (1)$$

in which p is the chance (probability), $f(\underline{x})$ the **Joint Probability Density Function** (JPDF) of the vector of random variables $\underline{X}$, such as material parameters, dimensions and loads. $G(\underline{x})$ is called the **limit-state function** (also referred to as **failure mode** or **performance function**) and is defined such that failure occurs when $G(\underline{x})$ becomes less than zero (failure domain). The domain where $G(\underline{x})$ equals zero is called the **limit-state**.
This is a very general expression, which is applicable to every type of stochastic problem. The objective of a stochastic analysis is to obtain the solution of this equation, i.e. solving a multi-dimensional integral equation. This, in general, is not an easy task due to the following reasons:

- The integral equation is multi-dimensional, depending on the number of random variables.
- In general, the joint probability density function f($\underline{x}$) will be unknown in explicit form. If known, it consists of complex functions, e.g. exponential and power functions.
- In general, the limit-state function G($\underline{x}$) will be unknown in explicit form and often is rather complicated, requiring for example a finite element analysis.

In order to obtain a solution of this equation a number of stochastic methods are available, such as: direct integration, Monte-Carlo simulations, importance sampling, first- and second-order reliability methods.

For these numerical stochastic methods the limit-state function G($\underline{x}$) must be evaluated for several values of the random variables, which can be very time consuming. Such a limit-state function evaluation will be called a **function evaluation** throughout this paper. Basically, all the numerical stochastic methods differ from each other in the way they perform the function evaluations, i.e. which values of the random variables are chosen for the next function evaluation. In general, the more complex the algorithm that is used, i.e. the more knowledge of the characteristics of the problem is implemented in the algorithm, the less function evaluations are needed to obtain a reliable solution.

For our purpose (structure reliability) we are interested in a method which is able to determine small probabilities requiring as less function evaluations as possible, because the probability values of interest for aircraft structural components (probability of failure) are rather small (in general less than 10^{-4}) and the limit-state is known only implicitly requiring considerable computer time to evaluate (finite element software, crack growth programs).

For example, in case of the very well known and widely used Monte-Carlo method the number of simulations (function evaluations) required to obtain a reliable result for probabilities of the order of 10^{-4}, would be of the order 10^{6}, which is very unrealistic in case evaluations require finite element analyses.

In the past 20 years more efficient stochastic methods, than the Monte-Carlo method, have been developed. Among them are the **First-order Reliability Method (FORM)** and **Second-order Reliability Method (SORM)**. Which are here the most promising methods and are explained in more detail. SORM is an extension of FORM resulting in a higher accuracy. The number of function evaluations of these methods strongly depends on the scheme used to find the so-called most probable point, which is the main subject of this paper.

Both, FORM and SORM, as well as the Monte-Carlo method are implemented in the computer program RAP (Reliability Analysis Program). In chapter 1 the FORM method is discussed and in chapter 2 the SORM method. In chapter 3 the capability and accuracy of these stochastic methods and the correct implementation is demonstrated by analysing two problems.

1 FIRST-ORDER RELIABILITY METHOD (FORM)

1.1 Introduction

In general, the solution of the multi-dimensional integral (1) is complicated. However, provided some special conditions have been satisfied this multi-dimensional integral can be reduced to a one-dimensional integral equation with known solution. These restrictions are:

1) All the random variables have a standard normal distribution function
2) All the random variables are independent of each other.

   In this case the joint probability density function becomes a multiplication of the probability density functions of the separate random variables, which are known functions.

$$f(\underline{x}) = f(x_1, x_2, x_3, ..) = f(x_1)\, f(x_2)\, f(x_3)..$$

3) The limit-state is a hyper plane.
   Which means that the limit-state is a linear combination of the random variables. In two-dimensions (two random variables) the limit-state is represented by a line. In three dimensions by a plane, etc.

On first sight these restrictions seems very tight and not very realistic, however the first two restrictions can be overcome by applying appropriate transformations to the problem, i.e. transforming a set of dependent random variables into a set of independent variables and transforming non-normal variables into standard normal variables (see sections 1.3 and 1.4).

The third restriction implies that the real limit-state is approximated by a hyper plane, in

two dimensions a straight line, i.e. linear function or first-order polynomial, at the location of the most probable point. The method therefore is called a first-order method. For a broad range of problems this is a reasonable approximation. However, if the limit-state has a large curvature this approximation causes a considerable error. One way to reduce this error is by increasing the order of the polynomial approximation (SORM).
Having satisfied the above three conditions it can be proven that the multi-dimensional integral equation transforms into a one-dimensional integral with known solution, that is:

$$p = P(G \leq 0) = \int_{G(\underline{x})\leq 0} f(\underline{x})\, d\underline{x} = \Phi(-\beta) \qquad (2)$$

where Φ is the standard normal Cumulative Distribution Function (CDF) and β the so-called **safety index** or **most probable point (MPP)**. The last name reflects the fact that β can be interpreted as being the point on the limit-state with the smallest distance to the origin of the JPDF (consisting of independent standard normal distributions), being the point with the highest probability, i.e. most probable point at which the structure will fail. Figure 2 gives the graphical interpretation of β (MPP) for the two dimensional case.
In this way the stochastic problem is reduced to finding the point on the limit-state ($G(\underline{x})=0$) with the smallest distance to origin, which is nothing else then a minimalisation problem. Solving the multi-dimensional integral equation is replaced by solving an optimisation problem. The optimisation problem yields:

$$\begin{aligned} &\text{Minimise:} && \sqrt{\underline{x}^T \underline{x}} && \text{(Objective)} \\ &\text{Subjected to:} && G(\underline{x}) = 0 && \text{(Constraint)} \end{aligned} \qquad (3)$$

There are numerous ways in which this minimalisation problem can be solved. A few are discussed in [8]. In this paper another very stable and efficient approach will be discussed in more detail.

1.2 Determination of most probable point

1.2.1 Introduction

In section 1.1 it has been explained how the stochastic problem defined by the integral equation (1) can be reformulated into a first-order minimalisation problem of equation (3), provided some specific requirements have been fulfilled. The method to solve this optimisation problem is discussed in this section. The objective of the optimisation problem is finding the most probable point (MPP), i.e. the point with the smallest distance to the origin of the JPDF.
Hasofer and Lind [6] are the founders of the FORM method in crude form. They defined the safety index/MPP β and assumed that all the variables are standard normal distributed. Rackwitz and Fiessler [9] suggested a simple iterative scheme with which this safety index or most probable point could be determined. Due to its simplicity, it is a widely used approach, but has the disadvantage that the method does not always converge and requires a considerable amount of function evaluations. Therefore, a better optimisation scheme has to be applied. Liu and Der Kiureghian [8] analysed a number of optimisation schemes of which some showed good convergence for various problems, but lacked efficiency. Therefore, another approach is adopted here, which is known as the **approximate model concept**. The idea behind it is fairly simple. Before explanation of this method first a number of terms are introduced in order to generalise the idea.

1.2.2 Approximate model concept

Equation (3) states the optimisation problem to be solved. The function, which has to be minimised, is called the **objective** and the function to which this objective is subjected is called a **constraint**. In general, optimisation problems can have multiple constraints. In view of a stochastic problem this means that there are multiple limit-states. An example is failure of a structure due to different failure mechanisms. The multiple limit-state situation introduces some new aspect to the problem which were not analysed in detail during this work and are therefore not discussed any further in this paper. Information about this topic can be found in references [4] and [7].

In the approximate model concept the objective and constraints are approximated by Taylor polynomials to form an approximate model. In other words the original functions, which can be complex and time consuming to evaluate, are replaced by simple explicit polynomial relations. Evaluation of these relations require nearly no computing time at all and therefore can be easily evaluated a great number of times. With these

approximations an optimisation cycle, requiring many function evaluations, is performed resulting in an optimum, in our case a value for the MPP. However, this optimum is based on an approximation of the original objective and constraints and thus will not be the exact optimum of the original problem. Therefore, a new approximate model is formed around the new optimum found and a new optimisation cycle is performed. Both steps, construction of the approximate model and optimising it, are called a **maxi-cycle**. Repeated maxi-cycles are performed until one of the convergence criteria has been met or a maximum number of cycles have been exceeded.

This approach has a number of advantages. First of all the optimisation algorithm used is not critical any more (from a convergence point of view), because the approximate model is a well behaving optimisation problem. Secondly, it reduces the total amount of function evaluations of the original function, since the approximate model is a reasonable approximation for a larger area around the point in which the functions are approximated.
In order to obtain the coefficients of the Taylor polynomials a number of function evaluations of the original functions are necessary. The computing time now is spend on constructing the approximate model requiring less function evaluations than optimising the original problem directly.

1.2.3 Approximate model

The objective (3) is a very simple explicit function of the random variables requiring nearly no computation time to evaluate it. For this reason it is not approximated. The Taylor approximation of the limit-state function has the following form:

$$G(\underline{x}) = G(\underline{x}_a) + \nabla G(\underline{x}_a)\,\underline{dx} + \frac{1}{2}\,\underline{dx}^T H\,\underline{dx} + HOT$$

$$\underline{dx} = \underline{x} - \underline{x}_a \qquad (4)$$

where $\underline{x}$ is a vector of random variables, $\underline{x}_a$ the point around which the function is approximated. ∇G is a vector containing the gradients of the limit-state function to its variables, H is the Hessian matrix containing second-order derivatives of G and HOT are the Higher Order Terms containing higher-order derivatives.

In order to obtain a second-order Taylor approximation, the original limit-state function G needs to be evaluated a number of times. First of all G needs to be evaluated at the approximation point $\underline{x}_a$. Secondly, to determine the gradients of G a number of evaluations are needed depending on the method used and finally, the Hessian matrix consisting of second-order derivatives of G need a considerable amount of function evaluations.
The gradients of G (first-order derivatives) are determined with a finite difference scheme. Three different schemes have been implemented: backward, forward and central. The backward and forward scheme require one evaluation for every derivative plus the result of the already performed function evaluation at the point function $\underline{x}_a$. The central scheme requires an extra function evaluation for every variable and is thus less efficient than the other two schemes, without, in general, resulting in a higher convergence rate.
The elements of the Hessian matrix consist of second-order derivatives of G to its variables. Therefore, evaluating this matrix requires a considerable amount of function evaluations. This is due to its many elements and due to the evaluation of the second-order derivatives, which requires more function evaluations than a first-order derivative. One is therefore very tempted to neglect this Hessian matrix and proceed with only a first-order Taylor approximation of the limit-state. However, the convergence of the overall problem becomes worse and sometimes the problem does not converge at all. A solution to this problem is to approximate this Hessian matrix, i.e. its second-order derivative, by using the values of the gradients and variables at the point x_a of the current and previous point (previous maxi-cycle). The second-order Taylor approximation, called **Newton method**, is then called a **quasi-Newton method** or a **variable metric method**. The approximate Hessian matrix is updated every maxi-cycle at the new optimal point x_a. Many of such update formulas are available, but the most popular is the so-called Broydon-Fletcher-Goldfarb-Shanno (BFGS) method [12].

1.2.4 Optimisation of the approximate model

As explained in section 1.2.2 a maxi-cycle consists of construction of the approximate model followed by an optimisation of this approximate model. In the previous section it was explained how the approximate model is constructed. In this

section the topic of interest is the optimisation of this model.
There are many optimisation algorithms available to solve this problem, see for example reference [12]. As mentioned before, one of the advantages of the approximate model concept was the resulting relatively simple and easy to solve optimisation problem. The choice of optimisation algorithm is therefore not very critical. Instead of implementing an algorithm, it was chosen to use an existing optimisation code called DOT [13]. The program is based on the modified feasible directions algorithm. It's outside the scope of this paper to extensively explain the working of DOT. For details the interested reader is referred to the appendix E of [13].
Roughly speaking, the DOT algorithm is based on finding a feasible (no constraints are violated) search direction at the current point in the domain, in which the algorithm tries to search for a better optimum. Once a better optimum is found the process of determining a new search direction and finding an even better optimum is repeated at this new point.
In order to determine a feasible search direction at the current point $\underline{X}$ (vector in multi-dimensional case) DOT needs the gradients of the objective and constraints at the current point. A direction is called feasible when no constraints are violated in the direct neighbourhood of the current point.
Since we are optimising the approximate model, the gradients required are the gradients of the approximated model. In our case this implies the gradients of the objective, which are exact since the objective has not been approximated, and the gradients of the approximated limit-state function.
The gradients of the objective can be determined by:

$$\frac{\partial \text{obj}}{\partial x_i} = \frac{x_i}{\sqrt{\underline{x}^T \cdot \underline{x}}}$$

The gradients of the approximate limit-state (4) can be determined by:

$$\nabla G(\underline{x}) = \nabla G(\underline{x}_a) + H\, d\underline{x}$$

After a feasible search direction has been obtained a line search is performed in that direction to reach a better optimum. During this line search DOT will ask for function values (in the approximate space) at specific points. This process of finding a feasible search direction and performing a line search is repeated until no further improvement of the optimum can be found or some convergence criteria have been met.

1.2.5 Special features

In order to improve the convergence of the overall process the following special features have been applied.

1) The random variables are scaled to ensure that their values lie in the range $-1 \le X_i \le 1$
2) The limit-state function is scaled such that $G(x) \le 1$
3) DOT is unable to optimise a highly infeasible problem, when a constraint is violated, i.e. where the limit-state has a positive value. The more the point lies away from the limit-state in the domain where it is positive, the more infeasible the problem becomes.

   This situation can be overcome by reformulating the problem into an unconstraint optimisation problem with an objective defined in terms of the constraints:

$$\text{obj}(\underline{x}) = \frac{1}{\rho} \ln\left(\sum_{i=1}^{\text{nconst}} e^{\rho\, G_i(\underline{x})} \right)$$

   with ρ being a large constant and nconst the number of constraints.
   The derivatives of this new objective yield:

$$\frac{\partial \text{obj}}{\partial x_i} = \frac{1}{\sum_{j=1}^{\text{nconst}} e^{\rho\, G_j(x)}} \sum_{j=1}^{\text{nconst}} \left(e^{\rho\, G_j(x)} \frac{\partial G_j(x)}{\partial x_i} \right)$$

   Both the objective and its derivatives depend only on known values and thus require no new function evaluations. For the cases throughout this paper only one limit-state is present and the above expressions simplify to the original constraint value and its derivative.
   With this new problem an optimisation is performed which results in a point in the feasible domain, which is used as the next point in the original optimisation problem.
4) As starting point of the optimisation process the origin of the optimisation domain is chosen for the following two reasons. Firstly, the objective is a function with one global minimum located at the origin. Because it is unknown on forehand where the limit-state is located, it is better to choose the starting point at this minimum. Otherwise, a less favourable point on the wrong side can be selected, decreasing the convergence rate. Secondly, for the same reason the chance on selecting an initial point in the feasible domain is very small. The initial problem therefore is nearly

always infeasible when a starting point is selected randomly. For such problems special measures have to be taken (see point 3), which is not the case when the point is located at the origin, because at this point the value of the objective and its derivatives are zero and therefore do not take part in the first maxi-cycle, i.e. for a single limit-state problem a similar situation is achieved as described under point 3.

5) As explained in section 1.2.2, the optimum found with the approximate model after a maxi-cycle is not the exact optimum and a number of maxi-cycles are necessary to locate the real optimum. However, due to the fact that the Hessian is approximated (see section 1.2.3) the error in the location of the optimum after the first few maxi-cycles can be large. To prevent in these cases an oscillating solution process it is more efficient to restrict the optimisation domain around the current point. This is done with so-called **move-limits**.
6) The final optimum found will not be located exactly on the limit-state. The optimisation process will try to prevent the situation where a point is exactly located on a constraint, in order to prevent a cross over of the point in the next cycle leading to an infeasible situation. Therefore, the optimum found is not the exact minimum distance (MPP) and the limit-state value not exactly zero. The MPP can be corrected for this non-zero limit-state value using (4), as follows. Knowing that $G(x_\beta) = 0$ and furthermore assuming that:
   - The term with the Hessian can be neglected. This means that with a linear approximation of the limit-state function around the approximated optimum x_a the exact optimum x_β can be described sufficiently accurate. This condition is fulfilled because both points lie very close to one another. This condition yields with (4):
   $$0 = G(\underline{x}_a) + \nabla G(\underline{x}_a)\,(\underline{x}_a - \underline{x}_\beta)$$
   - x_a points in the correct direction, i.e. the vector is normal to the limit-state but only the length of the vector not equals the exact beta value. This condition is normally also fulfilled by the optimisation process. This condition yields:
   $$\underline{x}_a - \underline{x}_\beta = \alpha\ \underline{x}_a$$

   where α is the correction factor searched.
   Combining both equations, of which all the values are already known, finally result in:

$$\alpha = -\frac{G(\underline{x}_a)}{\nabla G(\underline{x}_a)\cdot \underline{x}_a}$$

and finally with this the corrected β becomes:

$$\beta = |x_a| + \alpha$$

1.3 Non-normal to standard normal variables transformation

In section 1.1 the three conditions to which the stochastic problem is restricted, in order to use the solution procedure described in this chapter, were described. Already mentioned is that all three restrictions can be overcome in an approximate sense such that the method can be successfully applied to a much broader class of problems. In this section a method to deal with the first restriction of section 1.1, i.e. the random variables must be standard normally distributed, is presented. A first approach was suggested by Ditlevsen [5] and later on extended by Chen and Lind [2].
Here, the exact transformation is applied. Let X have an arbitrary cumulative distribution function (CDF) F(x), which is the integral of the PDF f(x).

$$p = F(x)$$

The probability values of X lie in the range $0 \le p \le 1$. A certain probability value can be represented by any type of distribution function, although at another location referred to as Z. In our case we select a standard normal distribution. The variable X hereby is transformed from the X-space of the original distribution function to the standard normal space Z. Mathematically formulated as:

$$F(x) = \Phi(z)$$

where F(x) represents the original cumulative distribution function of X and $\Phi(z)$ the standard normal distribution function.
In other words, knowing a value for X one can find a value for Z in the standard normal space by:

$$z = \Phi^{-1}(F(x))$$

where $\Phi^{-1}()$ is the inverse of $\Phi()$, or vice versa.
In this way any non-normally distributed variable can be transformed to the standard normal space or vice versa, although the transformation is non-linear.

1.4 Dependent to independent variables transformation

A method to deal with the second restriction of section 1.1, i.e. the random variables have to be independent of each other, is given in [14].
In equation (1) the basic stochastic problem was defined and the JPDF was introduced. Rosenblatt [10] has introduced a transformation by which the non-normal dependent random variables are transformed to independent standard normal variables. With this transformation the first two restrictions would have been resolved at once. However, this transformation requires that the JPDF is known, which is hardly ever the case for any real life problem. The Rosenblatt transformation is therefore more an academic then a practical solution. There exist a more practical, although approximate, solution, the so-called **Nataf** model. Wu [14] has suggested an approach for solving this Nataf model.

2 SECOND-ORDER RELIABILITY METHOD (SORM)

2.1 Second-order probability formula's

In the previous chapter the First-Order Reliability Method (FORM) has been explained in detail and it was shown how the first two requirements of section 1.1 could be fulfilled in a general sense. This chapter is concerned with the third requirement and how, in general, FORM can be extended in order to obtain a higher accuracy. In section 1.1 it has been explained that the third requirement can be viewed at as a linear approximation of the limit-state: G(x)=0 and that this approximation introduces a considerable error in cases where the limit-state is far from linear. A higher-order approximation of the limit-state will give a considerable better accuracy in these cases, an example will be shown in chapter 3. In general, one can imagine that the limit-state surface is non-linear.
Equation (4) is no longer valid for a second-order limit-state. Breitung [1] was the first who came up with an approximate solution for this case, which is a correction on the FORM solution (2):

$$p = P(G \le 0) = \Phi(-\beta) \prod_{i=1}^{n-1} \frac{1}{\sqrt{(1+\beta\,\kappa_i)}}$$

where β again is the MPP and κ_i are the main curvatures of the limit-state surface at β, taken positive when convex. This solution becomes exact as $\beta \to \infty$.
More accurate formulas have been developed by Tvedt [11]. These are in order of increasing accuracy:

- Tvedt's **three-term formula** (TT)
- Tvedt's **single-integral formula** (SI):
- Tvedt's **double-integral formula** (DI):

This last integral is exact when all curvatures are positive.

2.2 Determination of curvatures

The formulas introduced in the previous section require the determination of the main curvatures of the limit-state surface at the MPP. The main curvatures are based on the eigenvalues of H.
In section 1.2.3, explaining the approximate model, it was shown that calculation of the Hessian matrix is very expensive. Therefore, several methods have been proposed to approximate the Hessian.
The most common method to calculate the second-order derivatives at the MPP is a finite difference scheme.
Another method, introduced by Der Kiureghian [3], the curvatures in the different directions are determined by fitting a paraboloid through a number of selected points on the limit-state surface at some distance from the MPP. This method has a number of advantages over finite difference methods [3] and is used here.

3 APPLICATIONS

3.1 Introduction

The second-order method, and as a result of that also the first-order method, explained in the previous chapters have been implemented in a computer program called **RAP (Reliability Analysis Program)**.
In this chapter two numerical examples are analysed to demonstrate the:

1) Correctness of the implemented computer code
2) Accuracy of the method
3) Performance of the computer code (number of function evaluations)
4) Capabilities of the software

The first example concerns the stochastic static deflection of a beam due to variability in the elasticity modulus and applied force. In the second example the influence on the dynamical behaviour of a square simply supported plate due to the variability in the plate thickness and its elasticity modulus is examined.
In general, the number of function evaluations required by RAP to obtain a FORM solution is given by:

$$N_{FORM} = N_{maxcycle}\,(N_{var} + 1) + 1$$

where N_{var} is the number of random variables in the problem and $N_{maxcycle}$ is the number of maxi-cycles to reach an optimum MPP, usually 3 to 4.
The extra function evaluations required for determination of the curvatures to obtain a SORM solution is given by:

$$N_{curvs} = 2\,(N_{var} - 1)\,N_{secant}$$

where N_{secant} stands for the number of iterations required by the secant algorithm to reach a converged solution, usually 3 to 4.

3.2 Stochastic static problem

In this section a simple static problem is discussed to show the application of the stochastic method. The problem, figure 1, consists of a prismatic beam of length **L** with moment of inertia **I**, clamped on one side and simply supported on the other side and loaded by a point force **F** at point **C** on a distance **a** from the clamping. Our interest lies in the maximum deflection of the beam at the loading point δ_c and its statistical behaviour due to the random behaviour of the force **F** and elasticity modulus **E** of the beam.

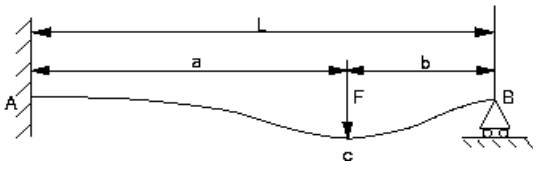

Figure 1. Definition static beam problem.

The deflection of the beam can analytically be determined by means of:

$$\delta_c = \frac{Fa^2b}{2EI}\left[\left(\frac{2a+3b}{6L}\right)\left(3-\left(\frac{b}{L}\right)^2\right)-1\right]$$

The following parameter values were used:

L = 3 m
a = 2 m
b = 1 m
I = $1.7\ 10^{-6}$ m^4
E = Log-normally distributed:
$\mu = 7\ 10^{10}$ N/m^2 $\sigma = 7\ 10^{9}$ N/m^2
F = Extreme value distribution type I max:
$x_0 = 995$ N $\alpha = 78$ N

Suppose the following question was raised: *what is the probability that the deflection at point-C is larger than 3 mm?*
This question can be answered by RAP by applying the following limit-state function:

$$G(E, F) = 3 - \delta_c(E, F)$$

Figure 2 depicts this limit-state G=0 for this case. The MPP, the point having the closest distance to the origin in standard normal space U, is also shown and yields:

$$\beta = 2.4357$$

With this β-value and equation (2) the FORM probability value becomes:

$$p_{FORM} = 7.43\ 10^{-3}$$

From figure 2 it is clear that the limit-state is not linear (straight line). The SORM result (based on the Tvedt's DI formula [11]) correcting for this, yields:

$$p_{SORM} = 8.82\ 10^{-3}$$

which is a higher value than predicted by FORM, consistent with the shape of the limit-state.

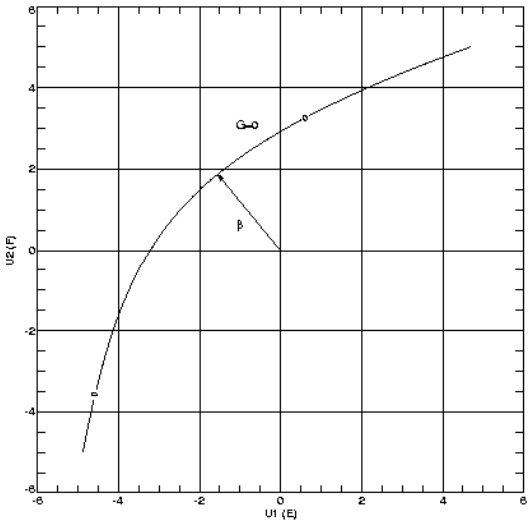

Figure 2. Two-dimensional limit-state G(E,F)=0 for beam problem.

Also a Monte-Carlo analysis, also included in RAP, was performed with 10^6 simulations resulting in a probability value of:

$$p_{MC} = 8.85\ 10^{-3}$$

with probability bounds:

$$8.63\ 10^{-3} < p_{MC} < 9.07\ 10^{-3}$$

This confirms the correct implementation of the methods.
In order to obtain the FORM solution, 13 function evaluations were required including one to obtain the value of the limit-state at the final MPP. Another 8 function evaluations were required to obtain the SORM solution, leading to a total of 21, in contrast to the large amount (10^6) required to obtain a reliable Monte-Carlo solution!

3.3 Stochastic dynamic problem

The second example problem concerns the dynamical behaviour of a square simply supported plate of length **a,** width **b** and thickness **t**. The material is characterised by its elasticity modulus **E**, density **ρ** and Poison's constant **ν**. Assumed here is that both the thickness and elasticity modulus are normally distributed.
The following parameter values were used:

- a = 2 m
- b = 1 m
- t = Normally distributed:
  $\mu = 1\ 10^{-3}$ m $\sigma = 1\ 10^{-4}$ m
- E = Normally distributed:
  $\mu = 7\ 10^{10}$ N/m^2 $\sigma = 7\ 10^{9}$ N/m^2
- ρ = 2800 kg/m^3
- ν = 0.3

Suppose the plate is part of a system which is excitated by an external source at a frequency of 1.5 Hz. If an eigenfrequency of the plate would lie close to the excitation frequency this would cause considerable vibrations of the plate, which has to be avoided.
A deterministic analysis of the plate, with mean values for t and E, would yield a first eigenfrequency of 2.97 Hz sufficiently away from the excitation frequency, so no problems should be expected. However, in reality for example t and E are stochastic, having a normal distribution with a coefficient of variation around the 10%.
If t and E are normally distributed variables: *what would be the chance that the eigenfrequency of the system would lie at or in the close neighbourhood of the excitation frequency?*

This can be determined by constructing part of the CDF around the 1.5 Hz. Let's start by determining the probability that the first frequency of the plate is less than 1.6 Hz, represented by the limit-state:

$$G(t, E) = f_1(t, E) - 1.6$$

The eigenfrequency is determined by means of a finite element analysis using 10*10 quadratic shell elements.
Figure 3 depicts the limit-state G=0. The MPP yields:

$$\beta = 4.4235$$

With this the FORM probability value (2), requiring 9 function evaluations, becomes:

$$p_{FORM} = 4.86\ 10^{-6}$$

From figure 2 it is clear that the limit-state surface is not linear. The SORM result, based on the Tvedt's DI formula requiring an extra 8 function evaluations, correcting for this, yields:

$$p_{SORM} = 5.60\ 10^{-6}$$

This is a very small probability, which indicates that there is little risk that the frequency of the plate will lie in the neighbourhood of the excitation frequency.

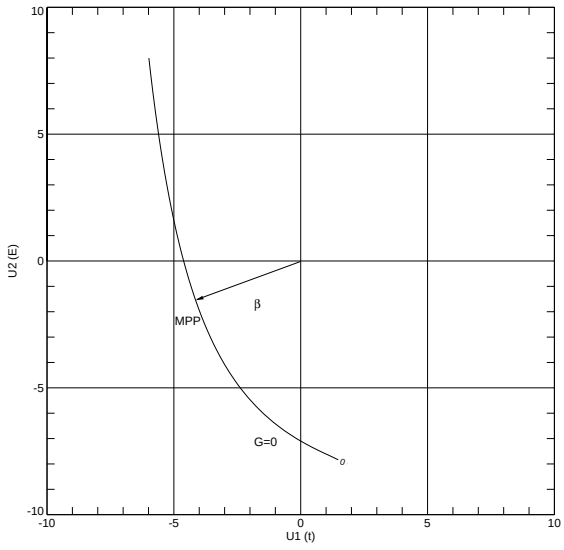

Figure 3. Two-dimensional limit-state (G(t,E)=0)

No Monte-Carlo results were determined, because the low probability would require at least 10^8 simulations, in contrast with the 13 function

evaluations required by the current stochastic method.

Figure 4 shows the left tail of the CDF constructed by computing the probability value for a number of other frequency values.

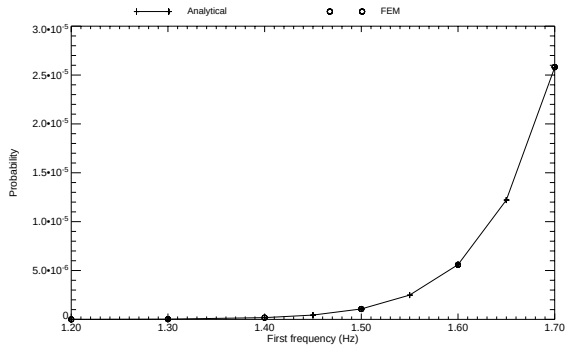

Figure 4. Calculated left-tail of CDF (SORM)

CONCLUSIONS

In this paper a very efficient stochastic method for probabilistic analysis has been described. The method has been implemented in an in-house computer code RAP.
The accuracy, correct implementation, performance and capabilities of the method are examined by analysing various example problems of which two are presented here, some more academic and some more realistic of nature. From these analyses, it can be concluded that:

- the various aspects of the method are implemented correctly,
- the method has a high accuracy for all the analysed problems,
- the performance of the method is very good, requiring only very few function evaluations,
- the method can be applied to a large variety of problems. No restrictions have been found so far.

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