



NLR-TP-2006-696

Stochastic differential equations on hybrid state spaces

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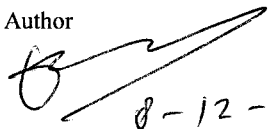
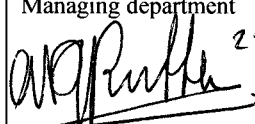
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This report contains a paper to appear in Stochastic Hybrid Systems: Recent Developments and Research Trends, Editors: Christos G. Cassandras and John Lygeros, CRC Press.

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Customer: National Aerospace Laboratory NLR
Working Plan number: 2005 AT.1.A
Owner: National Aerospace Laboratory NLR
Division: Air Transport
Distribution: Unlimited
Classification title: Unclassified
September 2006

Approved by:

Author  8-12-06	Reviewer Anonymous peer reviewers	Managing department  22/12/06
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Chapter 2

Stochastic Differential Equations on Hybrid State Spaces

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2.1 Introduction

In studying a wide variety of real-world phenomena we usually encounter processes the course of which cannot be predicted beforehand. For example: sudden deviation of the altitude of an aircraft from a prescribed flight level; reproduction of bacteria in a favorable environment; movement of a stock price on a stock exchange. Such processes can be represented by stochastic movement of a point in a particular space specially selected for each problem. The proper choice of the phase space turns physical, mechanical, or any other real-world system into a dynamical system (it means that the current state of the system determines its future evolution). Similarly, by a proper choice of the phase space (or state space) an arbitrary stochastic process can be turned into a Markov process, i.e., a process the future evolution of which depends on the past only through its present state. This property is called the Markov property. From a whole set of stochastic processes this Markov property

singles out a class of Markov processes for which powerful mathematical tools are available.

Continuous time Markov processes have been successfully used for years in stochastic modelling of various continuous time real-world dynamical systems with either Euclidean or discrete valued phase spaces. Recently, there is a great interest in more complex continuous time stochastic processes with components being hybrid, i.e., containing both Euclidean and discrete valued components. Such processes are called stochastic hybrid processes. Euclidean and discrete valued components may interact, i.e., Euclidean valued components may influence the dynamics of discrete valued component and vice versa. This makes the modelling and the analysis of stochastic hybrid processes quite involved and challenging. Several classes of stochastic hybrid processes have been studied in the literature, e.g., counting processes with diffusion intensity [21, 17], diffusion processes with Markovian switching parameters [22, 18], Markov decision drift processes [20], piecewise deterministic Markov processes [5, 6, 14], controlled switching diffusions [7, 8, 1], and more recent stochastic hybrid systems of [12, 19]. All these stochastic hybrid processes arise in various applications, have different degrees of modelling power, and have different properties inherent to the problems that they have been developed for.

There exist two directions in the development of theory of Markov processes: an analytical and a stochastic direction. Transition densities or transition probabilities are the starting point of the analytical Markov process theory. It studies various classes of transition densities and transition probabilities, which are described by equations (for example, by partial differential equations). When proving the existence of the corresponding Markov processes, any obtained conditions and properties on transition densities and probabilities are simply interpreted as certain properties of these processes. Broadly speaking, the approach taken by analytical Markov process theory could be compared with the analysis of the properties of random variables on the basis of their distribution functions or densities. In the stochastic theory a Markov process is constructed directly as a solution to a stochastic differential equation (SDE). The main advantage is that it is easier to study a Markov process as a solution of a particular equation than a Markov process that is implicitly defined through its transition density or probability. Moreover, the theory of SDE became a powerful tool for constructive description of various classes of stochastic processes including the processes which are semimartingales. Semimartingales form one of the most important and general class of stochastic processes which includes diffusion-type processes, point processes, and diffusion-type processes with jumps that are widely used for stochastic modelling. Considering SDE with semimartingale solutions gives an advantage. It allows the use of the powerful stochastic calculus available for the semimartingale processes when performing complex stochastic analysis. This has motivated many studies in the past to consider Markov processes that are solutions of SDE. However, most of the studies consider only Euclidean valued Markov processes and only a few of them treat SDE, the solutions of which are Markov processes with a hybrid state space. This chapter aims to give an overview of stochastic approaches of modelling hybrid state Markov processes as solutions to stochastic differential equations. In a series of recent studies, Blom

[2], Ghosh and Bagchi [9], and Krystul and Blom [15] developed distinct classes of stochastic hybrid processes as solutions of SDE on a hybrid state space. These classes have different modelling power and cover a wide range of interesting phenomena (see the first column of Table 2.1), though, all they contain, as a subclass, the switching diffusion processes of Ghosh *et al.* [8], described in detail in Chapter 5 of this volume.

Table 2.1: Combinations of features for various stochastic hybrid processes.

Features	[2], [9]	[3], [15]	[9]	[15]
Switching diffusion	✓	✓	✓	✓
Random hybrid jumps	✓	✓	-	✓
Boundary hybrid jumps	-	✓	✓	✓
Martingale inducing jumps	-	-	-	✓
Mode dependent dimension	-	-	✓	-

The features of stochastic hybrid processes in Table 2.1 are:

- Switching diffusion: between the random switches of the discrete valued component, the Euclidean valued component evolves as diffusion.
- Random hybrid jumps: simultaneous and dependent jumps and switches of discrete and Euclidean valued components are driven by a Poisson random measure.
- Boundary hybrid jumps: simultaneous and dependent jumps and switches of discrete and Euclidean valued components are initiated by boundary hittings.
- Martingale inducing jumps: the Euclidean valued components driven by a compensated Poisson random measure may jump so frequently that it is no longer a process of finite variation.
- Mode dependent dimension: the dimension of the Euclidean state space depends on the discrete valued component (i.e., the mode).

In the first part of the chapter we pay special attention to the modelling approach taken by Krystul and Blom [15]. Then we relate this to the models of Blom [2], Blom *et al.* [3], and Ghosh and Bagchi [9] and provide a comparison of these classes of stochastic hybrid systems.

This chapter is organized as follows. Section 2.2 provides a brief introduction to semimartingales. Section 2.3 presents the existence and uniqueness results for \mathbb{R}^n -valued jump-diffusions. Section 2.4 extends these results to hybrid state processes with Poisson and hybrid Poisson jumps [15]. In Section 2.5 we characterize a general

stochastic hybrid process which includes jumps at the boundaries [15]. Section 2.6 briefly describes stochastic hybrid models of Blom [2] and Ghosh and Bagchi [9] and compares various stochastic hybrid models. Finally, the Markov and the strong Markov properties for a general stochastic hybrid process [2], [15] are shown in Section 2.7.

2.2 Semimartingales and Characteristics

In this section, following [13], we provide basic results concerning semimartingales, their canonical representation, and their relation with the large class of SDE to be studied in this chapter.

Throughout this chapter we assume that a probability space (Ω, \mathcal{F}, P) is equipped with a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. The stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is called complete if the σ -algebra \mathcal{F} is P -complete and if every \mathcal{F}_t contains all P -null sets of \mathcal{F} . Note that it is always possible to “complete” a given stochastic basis, if it is not complete, by adding all subsets of P -null sets to \mathcal{F} and \mathcal{F}_t . We will therefore assume throughout this chapter that the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is complete.

The *predictable σ -algebra* is the σ -algebra \mathcal{P} on $\Omega \times \mathbb{R}_+$ that is generated by all left-continuous adapted processes (considered as mappings on $\Omega \times \mathbb{R}_+$). A process or random set that is \mathcal{P} -measurable is called *predictable*.

DEFINITION 2.1 The canonical setting. Ω is the “canonical space” (also denoted by $\mathbb{D}(\mathbb{R}^n)$) of all càdlàg (right-continuous and admit left hand limits) functions $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}^n$; X is the “canonical process” defined by $X_t(\omega) = \omega(t)$; $\mathcal{H} = \sigma(X_0)$; finally $(\mathcal{F}_t)_{t \geq 0}$ is generated by X and \mathcal{H} , by which we mean:

- (i) $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s^0$ and $\mathcal{F}_s^0 = \mathcal{H} \vee \sigma(X_r : r \leq s)$ (in other words, $(\mathcal{F}_t)_{t \geq 0}$ is the smallest filtration such that X is adapted and $\mathcal{H} \subset \mathcal{F}_0$);
- (ii) $\mathcal{F} = \mathcal{F}_{\infty-} (= \bigvee_t \mathcal{F}_t)$.

Throughout this chapter we assume that canonical setting of Definition 2.1 is in force. The \mathbb{R}^n -valued càdlàg stochastic process $\{X_t\}$ defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is a *semimartingale* if X_t admits a decomposition of the form

$$X_t = X_0 + A_t + M_t, \quad t \geq 0, \quad (2.1)$$

where X_0 is a finite-valued and \mathcal{F}_0 -measurable, $\{A_t\} \in \mathcal{V}^n$ is a process of *bounded variation*, $\{M_t\} \in \mathcal{M}_{loc}^n$ is an n -dimensional *local martingale* starting at 0, and for each $t \geq 0$, A_t and M_t are \mathcal{F}_t -measurable. Recall that $\{M_t\} \in \mathcal{M}_{loc}^n$ if and only if there exists a sequence of $(\mathcal{F}_t)_{t \geq 0}$ -stopping times $(\tau_k)_{k \geq 1}$ such that $\tau_k \uparrow \infty$ (P -a.s.)

for $k \rightarrow \infty$ and for each $k \geq 1$, the *stopped process*

$$\{M_t^{\tau_k}\} \text{ with } M_t^{\tau_k} = M_{t \wedge \tau_k}, k \geq 1, \quad (2.2)$$

is a *martingale*:

$$\mathbb{E}|M_t^{\tau_k}| < \infty, \mathbb{E}[M_t^{\tau_k} | \mathcal{F}_s] = M_s^{\tau_k} \text{ (} P\text{-a.s.)}, s \leq t. \quad (2.3)$$

Denote by $\mu = \mu(\omega; ds, dx)$ the measure describing the jump structure of $\{X_t\}$:

$$\mu(\omega; (0, t] \times B) = \sum_{0 < s \leq t} I_{\{\omega: \Delta X_s(\omega) \in B\}}(\omega), t > 0, \quad (2.4)$$

where $B \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$, i.e., the σ -algebra of Borel sets on \mathbb{R}^n , $\Delta X_s = X_s - X_{s-}$, and $I_{\{\omega: \Delta X_s(\omega) \in B\}}(\omega)$ is the indicator function of set $\{\omega : \Delta X_s(\omega) \in B\}$. By $\nu = \nu(\omega; ds, dx)$ we denote a compensator of μ , i.e., a predictable measure with the property that $\mu - \nu$ is a local martingale measure. This means that for each $B \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$:

$$(\mu(\omega; (0, t] \times B) - \nu(\omega; (0, t] \times B))_{t > 0} \quad (2.5)$$

is a local martingale with value 0 for $t = 0$.

A semimartingale $\{X_t\}$ is called *special* if there exists a decomposition (2.1) with a *predictable* process $\{A_t\}$. Every semimartingale with *bounded jumps* ($|\Delta X_t(\omega)| \leq b < \infty, \omega \in \Omega, t > 0$) is special [see 13, Chapter I, 4.24].

Let h be a truncation function, i.e., $\Delta X_s - h(\Delta X_s) \neq 0$ if and only if $|\Delta X_s| > b$ for some $b > 0$. Hence

$$\tilde{X}_t = \sum_{0 < s \leq t} (\Delta X_s - h(\Delta X_s)) \quad (2.6)$$

denotes the jump part of $\{X_t\}$ corresponding to *large jumps*. The number of the large jumps still is finite on $[0, t]$, for all $t > 0$, because for all semimartingales [13, Chapter I, 4.47]

$$\sum_{0 < s \leq t} (\Delta X_s)^2 < \infty, P\text{-a.s.} \quad (2.7)$$

The process $\{X_t - \tilde{X}_t\}$ is a semimartingale with *bounded jumps* and hence it is special:

$$X_t - \tilde{X}_t = X_0 + \tilde{B}_t + \tilde{M}_t \quad (2.8)$$

where $\{\tilde{B}_t\}$ is a predictable process and $\{\tilde{M}_t\}$ is a local martingale. The “tilde” above the process denotes the dependence on the truncation function h .

Every local martingale \tilde{M}_t can be decomposed as:

$$\tilde{M}_t = M_t^c + \tilde{M}_t^d \quad (2.9)$$

where M_t^c is a *continuous* (martingale) part and \tilde{M}_t^d is a *purely discontinuous* (martingale) part which satisfies:

$$\tilde{M}_t^d = \int_0^t \int h(x)(\mu(ds, dx) - \nu(ds, dx)). \quad (2.10)$$

Note that the continuous martingale part M_t^c does not depend on h . By definition of μ and $\{\tilde{X}_t\}$ we have

$$\tilde{X}_t = \int_0^t \int (x - h(x)) \mu(ds, dx). \quad (2.11)$$

Consequently, substitution of (2.9)–(2.11) into (2.8) yields the following canonical representation of semimartingale $\{X_t\}$:

$$X_t = X_0 + \tilde{B}_t + M_t^c + \int_0^t \int h(x) (\mu(ds, dx) - \nu(ds, dx)) + \int_0^t \int (x - h(x)) \mu(ds, dx). \quad (2.12)$$

Next we may assume $h(x) = x \cdot I_{\{x: |x| < 1\}}(x)$ and replace \tilde{B}_t by B_t . Then (2.12) takes on the form:

$$X_t = X_0 + B_t + M_t^c + \int_0^t \int_{|x| < 1} x (\mu(ds, dx) - \nu(ds, dx)) + \int_0^t \int_{|x| \geq 1} x \mu(ds, dx). \quad (2.13)$$

We denote by $\langle M_t^c \rangle$ the predictable quadratic variation of $\{M_t^c\}$, hence $(M_t^c)^2 - \langle M_t^c \rangle$ is a local martingale.

We call the *characteristics* associated with h of the semimartingale $\{X_t\}$ (if there may be an ambiguity on h) the triplet (B_t, C_t, ν) consisting of:

- (i) A predictable process $B_t = (B_t^i)_{i \leq n}$ in \mathcal{V}^n , namely the process $B_t = \tilde{B}_t$ appearing in (2.8);
- (ii) A continuous process $C_t = (C_t^{ij})_{i, j \leq n}$ in $\mathcal{V}^{n \times n}$, namely $C_t = \langle M_t^c \rangle$;
- (iii) A predictable random measure ν on $\mathbb{R}_+ \times \mathbb{R}^n$, namely the compensator of random measure μ associated to the jumps of X by (2.4).

DEFINITION 2.2 Jump diffusion. Let P be a probability measure on (Ω, \mathcal{F}) . Then $\{X_t\}$ is called a *jump diffusion* on $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, P)$ if it is a semimartingale with the following characteristics:

$$\begin{cases} B_t^i(\omega) = \int_0^t \alpha^i(s, X_s(\omega)) ds & (= +\infty \text{ if the integral diverges}) \\ C_t^{ij}(\omega) = \int_0^t \beta^{ij}(s, X_s(\omega)) ds & (= +\infty \text{ if the integral diverges}) \\ \nu(\omega; dt \times dx) = dt \times K_t(\omega, X_t(\omega), dx) \end{cases} \quad (2.14)$$

where:

$$\begin{cases} \alpha : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n & \text{is Borel} \\ \beta : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n & \text{is Borel, } c(s, x) \text{ is symmetric nonnegative} \\ K_t(\omega, x, dy) & \text{is a Borel transition kernel from } \Omega \times \mathbb{R}^n \times \mathbb{R}^n \\ & \text{into } \mathbb{R}^n, \end{cases}$$

with $K_t(\omega, x, \{0\}) = 0$.

Next, we relate the above with stochastic differential equations, partially following [13].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a stochastic basis endowed with:

- (i) $W = (W^i)_{i \leq m}$, an m -dimensional standard Wiener process (i.e., each W^i is a standard Wiener process, and the W^i 's are independent);
- (ii) p_i are Poisson random measures on $\mathbb{R}_+ \times U$ with intensity measure $dt \cdot m_i(du)$, $i = 1, 2$; here, (U, \mathcal{U}) is an arbitrary Blackwell space (one may take $U = \mathbb{R}^d$ for practical applications), and m_i , $i = 1, 2$, is a positive σ -finite measure on U, \mathcal{U} ; We denote the compensated Poisson random measure by $q_i(dt, du) = p_i(dt, du) - dt \cdot m_i(du)$, $i = 1, 2$.

Let us also be given the coefficients:

$$\begin{cases} a = (a^i)_{i \leq n}, & \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ b = (b^{ij})_{i \leq n, j \leq m}, & \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^m \\ f_1 = (f_1^i)_{i \leq n} & \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n \\ f_2 = (f_2^i)_{i \leq n} & \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n. \end{cases} \quad (2.15)$$

Let the initial variable be an \mathcal{F}_0 -measurable \mathbb{R}^n -valued random variable X_0 . The stochastic differential equation is as follows:

$$\begin{aligned} dX_t = a(t, X_t)dt + b(t, X_t)dW_t + \int_U f_1(t, X_{t-}, u)q_1(dt, du) \\ + \int_U f_2(t, X_{t-}, u)p_2(dt, du). \end{aligned} \quad (2.16)$$

Define two stochastic sets:

$$\begin{aligned} D_1 &= \{(\omega, t) : p_1(\omega; \{t\} \times U) = 1\}, \\ D_2 &= \{(\omega, t) : p_2(\omega; \{t\} \times U) = 1\}. \end{aligned}$$

If at least one of the Poisson random measures, p_1 or p_2 , has a ‘‘jump’’ at point (t, u) , then

$$\Delta X_t(\omega) = I_{D_1}(\omega, t) \cdot f_1(t, X_{t-}(\omega), u) + I_{D_2}(\omega, t) \cdot f_2(t, X_{t-}(\omega), u).$$

Next, let us assume that the following integrals make sense.

$$\int_0^t |a(s, X_s)| ds < \infty, \quad P\text{-a.s.} \quad (2.17)$$

$$\int_0^t \int_U |f_1(s, X_{s-}, u)|^2 ds m_1(du) < \infty, \quad P\text{-a.s.} \quad (2.18)$$

$$\int_0^t \int_U |f_2(s, X_{s-}, u)| p_2(ds, du) < \infty, \quad P\text{-a.s.} \quad (2.19)$$

$$\int_0^t |b^{ij}(s, X_s)|^2 ds < \infty, \text{ P-a.s. for any } i, j \in \{1, \dots, n\} \quad (2.20)$$

for every $t \in \mathbb{R}_+$. By a solution to the SDE (2.16) we mean a càdlàg \mathcal{F}_t -adapted process $\{X_t\}$ such that the following equation is satisfied with probability one for every $t \in \mathbb{R}_+$

$$\begin{aligned} X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s + \int_0^t \int_U f_1(s, X_{s-}, u) q_1(ds, du) \\ + \int_0^t \int_U f_2(s, X_{s-}, u) p_2(ds, du). \end{aligned} \quad (2.21)$$

If such process $\{X_t\}$ exists and conditions (2.17)–(2.20) are satisfied then it is a semimartingale with the characteristics, associated with truncation function $h = x \cdot I_{\{x: |x| < 1\}}(x)$, given by (2.14), where

$$\begin{aligned} \alpha(t, X_t(\omega)) &= \left[a(t, X_t(\omega)) - \int_{|f_1| \geq 1} f_1(t, X_{t-}(\omega), u) m_1(du) \right. \\ &\quad \left. + \int_{|f_2| < 1} f_2(t, X_{t-}(\omega), u) m_2(du) \right], \\ \beta(t, X_t(\omega)) &= b(t, X_t(\omega)) b^T(t, X_t(\omega)), \\ K_t(\omega, X_t(\omega), A) &= I_{D_1}(\omega, t) \cdot \int_U I_{A \setminus \{0\}}(f_1(t, X_{t-}(\omega), u)) m_1(du) \\ &\quad + I_{D_2}(\omega, t) \cdot \int_U I_{A \setminus \{0\}}(f_2(t, X_{t-}(\omega), u)) m_2(du). \end{aligned}$$

2.3 Semimartingale Strong Solution of SDE

There are two important notions of the sense in which a solution to stochastic differential equation can be said to *exist* and also two senses in which *uniqueness* is said to hold.

DEFINITION 2.3 Strong Existence. *We say that strong existence holds if given a probability space (Ω, \mathcal{F}, P) , a filtration \mathcal{F}_t , an \mathcal{F}_t -Wiener process W , two \mathcal{F}_t -Poisson random measures p_1, p_2 , and an \mathcal{F}_0 -measurable initial condition X_0 , then an \mathcal{F}_t -adapted process $\{X_t\}$ exists satisfying (2.21) for all $t \geq 0$.*

DEFINITION 2.4 Weak Existence. *We say that weak existence holds if given any probability measure η on \mathbb{R}^n there exists a probability space*

(Ω, \mathcal{F}, P) , a filtration \mathcal{F}_t , an \mathcal{F}_t -Wiener process W , two \mathcal{F}_t -Poisson random measures p_1, p_2 , and an \mathcal{F}_t -adapted process $\{X_t\}$ satisfying (2.21) for all $t \geq 0$ as well as $P(X_0 \in B) = \eta(B)$.

Strong existence of a solution requires that the probability space, filtration, and driving terms (W, p_1, p_2) be given first and that the solution $\{X_t\}$ then be found for the given data. Weak sense existence allows these objects to be constructed together with the process $\{X_t\}$. Clearly, strong existence implies weak existence.

DEFINITION 2.5 Strong Uniqueness. *Suppose that a fixed probability (Ω, \mathcal{F}, P) , a filtration $(\mathcal{F}_t)_{t \geq 0}$, an \mathcal{F}_t -Wiener process W , and two \mathcal{F}_t -Poisson random measures p_1 and p_2 are given. Let $\{X_t\}$ and $\{X'_t\}$ be two solutions of (2.16) for the given driving terms (W, p_1, p_2) . We say that strong uniqueness holds if*

$$P(X_0 = X'_0) = 1 \implies P(X_t = X'_t \text{ for all } t \geq 0) = 1, \quad (2.22)$$

i.e., $\{X_t\}$ and $\{X'_t\}$ are indistinguishable.

REMARK 2.1 Since solutions of (2.16) are càdlàg processes the requirement (2.22) can be relaxed to:

$$P(X_0 = X'_0) = 1 \implies P(X_t = X'_t) = 1, \text{ for every } t \geq 0. \quad (2.23)$$

■

DEFINITION 2.6 Weak Uniqueness. *Suppose we are given weak sense solutions*

$$\{(\Omega_i, \mathcal{F}_i, P_i), (\mathcal{F}_{i,t})_{t \geq 0}, \{X_{i,t}\}\}, i = 1, 2,$$

to (2.16). We say that weak uniqueness holds if equality of the distributions induced on \mathbb{R}^n by $X_{i,0}$ under P_i , $i = 1, 2$, implies the equality of the distributions induced on $\mathbb{D}(\mathbb{R}^n)$ by $\{X_{i,t}\}$ under P_i , $i = 1, 2$.

Strong uniqueness is also referred to as *pathwise uniqueness*, whereas weak uniqueness is often called *uniqueness in (the sense of probability) law*. Strong uniqueness implies weak uniqueness.

Next we present strong existence and strong uniqueness theorems for SDE (2.16). We assume that Wiener process W and Poisson random measures p_1 and p_2 are mutually independent. Suppose $\{W_t\}$, p_1 and p_2 are adapted to the given filtration $(\mathcal{F}_t)_{t \geq 0}$. If τ is a stopping time relative to \mathcal{F}_t and X_τ is an \mathcal{F}_τ measurable random variable, then we will be looking for an $\{\mathcal{F}_t\}$ -adapted process $\{X_t\}$, defined for

$t > \tau$, for which the following equation holds with probability 1

$$\begin{aligned} X_t = X_\tau + \int_\tau^t a(s, X_s) ds + \int_\tau^t b(s, X_s) dW_s + \int_\tau^t \int_U f_1(s, X_{s-}, u) q_1(ds, du) \\ + \int_\tau^t \int_U f_2(s, X_{s-}, u) p_2(ds, du). \end{aligned} \quad (2.24)$$

If equality (2.24) holds for all $t \in (\tau, \zeta)$, with ζ another stopping time, $\zeta > \tau$, then we will say that $\{X_t\}$ is the solution of SDE (2.16) on interval (τ, ζ) , if started at X_τ .

THEOREM 2.1 *A solution of Equation (2.16) for any given X_0 is strongly unique if the coefficients of Equation (2.16) satisfy the following conditions:*

(i) *for each $r > 0$ there exist a constant l_r , for which*

$$\begin{aligned} |a(s, x) - a(s, y)|^2 + |b(s, x) - b(s, y)|^2 \\ + \int_U |f_1(s, x, u) - f_1(s, y, u)|^2 m_1(du) \leq l_r |x - y|^2, \end{aligned}$$

for all $|x| \leq r$, $|y| \leq r$, $s \leq r$.

(ii) $\int_0^t \int_U |f_2(s, X_{s-}, u)| p_2(ds, du) < \infty$, *P*-a.s.,

(iii) $m_2(S_u) < \infty$, *where S_u is the projection on U of the support of $f_2(\cdot, \cdot, \cdot)$.*

PROOF See Theorem 3.8 in [15]. ■

Related to Theorem 2.1 is that two solutions of two different equations with equal initial conditions coincide as long as their coefficients coincide. We formulate this statement precisely, known as the theorem of local uniqueness.

THEOREM 2.2 *Suppose $\{X_t\}$ is a solution of Equation (2.21), and $\{\tilde{X}_t\}$ is a solution of Equation*

$$\begin{aligned} \tilde{X}_t = \tilde{X}_0 + \int_0^t \tilde{a}(s, \tilde{X}_s) ds + \int_0^t \tilde{b}(s, \tilde{X}_s) dW_s \\ + \int_0^t \int_U \tilde{f}_1(s, \tilde{X}_s, u) q_1(ds, du) + \int_0^t \int_U \tilde{f}_2(s, \tilde{X}_s, u) p_2(ds, du). \end{aligned}$$

If the conditions of Theorem 2.1 are satisfied and $a(s, x) = \tilde{a}(s, x)$, $b(s, x) = \tilde{b}(s, x)$, $f_k(s, x, u) = \tilde{f}_k(s, x, u)$ given $|x| \leq N$, then $X_s = \tilde{X}_s$ for $s \leq \tau$, where $\tau = \inf\{s : |X_s| \geq N\}$.

Next, we state the classical existence results for the following equation [11]:

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s + \int_0^t \int_U f_1(s, X_s, u) q_1(ds, du). \quad (2.25)$$

THEOREM 2.3 Assume that the coefficients of Equation (2.25) satisfy the following conditions:

- (i) $a(s,0)$, $b(s,0)$, $\int |f_1(s,0,u)|^2 m_1(du)$ are locally bounded with respect to s ,
- (ii) there exists increasing function $l(s)$ such that

$$|a(s,x) - a(s,y)|^2 + |b(s,x) - b(s,y)|^2 + \int_U |f_1(s,x,u) - f_1(s,y,u)|^2 m_1(du) \leq l(s)|x - y|^2.$$

Let us denote by \mathcal{F}_t the σ -algebra generated by X_0 , $q_1(ds, du)$, W_s with $s \leq t$. If X_0 is independent of W_s , $q_1(ds, du)$ and $\mathbb{E}|X_0|^2 < \infty$, then Equation (2.25) has \mathcal{F}_t -measurable solution, moreover $\mathbb{E}|X_s|^2 < \infty$.

THEOREM 2.4 Assume that for the coefficients of Equation (2.25) the following conditions hold:

$$|a(t,x)|^2 + |b(t,x)|^2 + \int_U |f_1(t,x,u)|^2 m_1(du) \leq l(1 + |x|^2),$$

and for any $r > 0$ one can specify constant l_r such that

$$|a(s,x) - a(s,y)|^2 + |b(s,x) - b(s,y)|^2 + \int_U |f_1(s,x,u) - f_1(s,y,u)|^2 m_1(du) \leq l_r |x - y|^2$$

for $s \leq r$, $|x| \leq r$, $|y| \leq r$. If X_0 is independent of $\{W_s, q_1(ds, du)\}$, and σ -algebras \mathcal{F}_t are constructed as in Theorem 2.3, then there exists an \mathcal{F}_t -measurable solution of (2.25) for every $t \in \mathbb{R}_+$.

REMARK 2.2 Suppose $\{\hat{\mathcal{F}}_t\}$ is some admissible filtration, and τ is a stopping time relative to this filtration. Let us consider the SDE for $t > \tau$:

$$X_t = X_\tau + \int_\tau^t a(s, X_s) ds + \int_\tau^t b(s, X_s) dW_s + \int_\tau^t \int_U f_1(s, X_s, u) q_1(ds, du). \quad (2.26)$$

Under conditions of Theorem 2.4, Equation (2.26) has $\hat{\mathcal{F}}_t$ -measurable solution, no matter what the $\hat{\mathcal{F}}_\tau$ -measurable variable X_τ is. To prove this, it suffices to consider the process \hat{X}_t which is a solution of the following equation.

$$\begin{aligned} \hat{X}_t = \hat{X}_0 + \int_0^t a(s + \tau, \hat{X}_s) ds + \int_0^t b(s + \tau, \hat{X}_s) d\hat{W}_s \\ + \int_0^t \int_U f_1(s + \tau, \hat{X}_s, u) \hat{q}_1(ds, du), \end{aligned} \quad (2.27)$$

where

$$\hat{W}_s = W(s + \tau) - W_\tau; \quad \hat{q}_1([s_1, s_2] \times du) = q_1([s_1 + \tau, s_2 + \tau] \times du). \quad (2.28)$$

Obviously, \hat{W} and \hat{q}_1 possess the same properties as W and q_1 , and are independent of \mathcal{F}_τ . Thus, for Equation (2.27), all derivations which were verified for Equation (2.25), hold as well, if expectations and conditional expectations with given X_0 are substituted by conditional expectation with respect to σ -algebra \mathcal{F}_τ . Obviously, then $X_t = \hat{X}_{t-\tau}$ will be the solution of Equation (2.26). ■

Now we state the existence theorem for general SDE (2.16).

THEOREM 2.5 *Assume that for Equation (2.16) the following conditions are satisfied:*

- (i) *The coefficients a, b, f_1 satisfy the conditions of Theorem 2.4.*
- (ii) *X_0 is independent of $\{W_s, q_1(ds, du), p_2(ds, du)\}$.*
- (iii) *Conditions (ii) and (iii) of Theorem 2.1 are satisfied.*

Let \mathcal{F}_t denote the σ -algebra generated by $\{W_s, q_1([0, s], du), p_2([0, s], du), s \leq t\}$ and X_0 . Then there exists an \mathcal{F}_t -measurable solution of Equation (2.16).

PROOF See Theorem 3.13 in [15]. ■

REMARK 2.3 The solution, whose existence was established in Theorem 2.5, is unique. Indeed, by Theorem 2.1 we have that for any enlargement of the initial probability space, any admissible filtration of σ -algebras $\tilde{\mathcal{F}}_t$, and any \mathcal{F}_0 -measurable initial variable X_0 , $\tilde{\mathcal{F}}_t$ -measurable solution of Equation (2.16) is unique. Since $\mathcal{F}_t \subset \tilde{\mathcal{F}}_t$, the solution X_t constructed in Theorem 2.5 will be also $\tilde{\mathcal{F}}_t$ -measurable, and therefore, there will be no other $\tilde{\mathcal{F}}_t$ -measurable solutions of Equation (2.16). ■

REMARK 2.4 The solution constructed in Theorem 2.5 is fully determined by the initial condition, Wiener process W and Poisson random measures p_1 and p_2 , i.e., it is a strong solution (solution-process). Thus, Theorem 2.5 states that there exists a strong solution of Equation (2.16) (strong existence), and from Remark 2.3 it follows that under conditions of Theorem 2.5 any solution of (2.16) is unique (strong uniqueness). ■

REMARK 2.5 Under the conditions of Theorem 2.5 the solution of SDE

(2.16) admits the decomposition (2.1) with

$$\begin{aligned} A_t &= \int_0^t a(s, X_s) ds + \int_0^t \int_U f_2(s, X_{s-}, u) p_2(ds, du) \in \mathcal{V}^n, \\ M_t &= \int_0^t b(s, X_s) dW_s + \int_0^t \int_U f_1(s, X_{s-}, u) q_1(ds, du) \in \mathcal{M}_{loc}^n, \end{aligned}$$

hence it is a semimartingale. ■

2.4 Stochastic Hybrid Processes as solutions of SDE

In this section we construct a switching jump diffusion $\{X_t, \theta_t\}$ taking values in $\mathbb{R}^n \times \mathbb{M}$, where $\mathbb{M} = \{e_1, e_2, \dots, e_N\}$ is a finite set. We assume that for each $i = 1, \dots, N$, e_i is the i -th unit vector, $e_i \in \mathbb{R}^N$. Note that the hybrid state space $\mathbb{R}^n \times \mathbb{M} \subset \mathbb{R}^{n+N}$ can be seen as a special subset of $(n+N)$ -dimensional Euclidean space. Let $\{X_t, \theta_t\}$ be an $\mathbb{R}^n \times \mathbb{M}$ -valued process given by the following stochastic differential equation of Ito-Skorohod type.

$$\begin{aligned} dX_t &= a(X_t, \theta_t) dt + b(X_t, \theta_t) dW_t + \int_{\mathbb{R}^d} g_1(X_{t-}, \theta_{t-}, u) q_1(dt, du) \\ &\quad + \int_{\mathbb{R}^d} g_2(X_{t-}, \theta_{t-}, u) p_2(dt, du), \end{aligned} \quad (2.29)$$

$$d\theta_t = \int_{\mathbb{R}^d} c(X_{t-}, \theta_{t-}, u) p_2(dt, du). \quad (2.30)$$

Here:

- (i) for $t = 0$, X_0 is a prescribed \mathbb{R}^n -valued random variable.
- (ii) for $t = 0$, θ_0 is a prescribed \mathbb{M} -valued random variable.
- (iii) W is an m -dimensional standard Wiener process.
- (iv) $q_1(dt, du)$ is a martingale random measure associated to a Poisson random measure p_1 with intensity $dt \times m_1(du)$.
- (v) $p_2(dt, du)$ is a Poisson random measure with intensity $dt \times m_2(du) = dt \times du_1 \times \bar{\mu}(d\underline{u})$, where $\bar{\mu}$ is a probability measure on \mathbb{R}^{d-1} , $u_1 \in \mathbb{R}$, $\underline{u} \in \mathbb{R}^{d-1}$ refers to all components except the first one of $u \in \mathbb{R}^d$.

The coefficients are defined as follows

$$\begin{aligned}
a &: \mathbb{R}^n \times \mathbb{M} \rightarrow \mathbb{R}^n \\
b &: \mathbb{R}^n \times \mathbb{M} \rightarrow \mathbb{R}^{n \times m} \\
g_1 &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R}^d \rightarrow \mathbb{R}^n \\
g_2 &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R}^d \rightarrow \mathbb{R}^n \\
\phi &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{M} \times \mathbb{R}^{d-1} \rightarrow \mathbb{R}^n \\
\lambda &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}_+ \\
c &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R}^d \rightarrow \mathbb{R}^N.
\end{aligned}$$

Moreover, for all $k = 1, 2, \dots, N$ we define measurable mappings $\Sigma_k : \mathbb{R}^n \times \mathbb{M} \rightarrow \mathbb{R}_+$ in a following manner

$$\Sigma_k(x, e_i) = \begin{cases} \sum_{j=1}^k \lambda(x, e_i, e_j) & k > 0, \\ 0 & k = 0, \end{cases} \quad (2.31)$$

function $c(\cdot, \cdot, \cdot)$ by

$$c(x, e_i, u) = \begin{cases} e_j - e_i & \text{if } u_1 \in (\Sigma_{j-1}(x, e_i), \Sigma_j(x, e_i)], \\ 0 & \text{otherwise,} \end{cases} \quad (2.32)$$

and function $g_2(\cdot, \cdot, \cdot)$ by

$$g_2(x, e_i, u) = \begin{cases} \phi(x, e_i, e_j, \underline{u}) & \text{if } u_1 \in (\Sigma_{j-1}(x, e_i), \Sigma_j(x, e_i)], \\ 0 & \text{otherwise.} \end{cases} \quad (2.33)$$

Let U_θ denote the projection of the support of function $\phi(\cdot, \cdot, \cdot, \cdot)$ on $\underline{U} = \mathbb{R}^{d-1}$. The jump size of X_t and the new value of θ_t at the jump times generated by Poisson random measure p_2 are determined by the functions (2.32) and (2.33) correspondingly. There are three different situations possible:

(i) Simultaneous jump of X_t and θ_t

$$\begin{cases} c(\cdot, \cdot, u) \neq 0 & \text{if } u_1 \in (\Sigma_{j-1}(x, e_i), \Sigma_j(x, e_i)], i, j = 1, \dots, N \text{ and } j \neq i, \\ g_2(\cdot, \cdot, u) \neq 0 & \text{if } u_1 \in (\Sigma_{j-1}(x, e_i), \Sigma_j(x, e_i)], i, j = 1, \dots, N \text{ and } \underline{u} \in U_\theta. \end{cases}$$

(ii) Switch of θ_t only

$$\begin{cases} c(\cdot, \cdot, u) \neq 0 & \text{if } u_1 \in (\Sigma_{j-1}(x, e_i), \Sigma_j(x, e_i)], i, j = 1, \dots, N \text{ and } j \neq i, \\ g_2(\cdot, \cdot, u) = 0 & \text{if } u_1 \in (\Sigma_{j-1}(x, e_i), \Sigma_j(x, e_i)], i, j = 1, \dots, N \text{ and } \underline{u} \notin U_\theta. \end{cases}$$

(iii) Jump of X_t only

$$\begin{cases} c(\cdot, \cdot, u) = 0 & \text{if } u_1 \in (\Sigma_{j-1}(x, e_j), \Sigma_j(x, e_j)], j = 1, \dots, N, \\ g_2(\cdot, \cdot, u) \neq 0 & \text{if } u_1 \in (\Sigma_{j-1}(x, e_j), \Sigma_j(x, e_j)], j = 1, \dots, N, \text{ and } \underline{u} \in U_\theta. \end{cases}$$

We make the following assumptions on the coefficients of SDE (2.29)–(2.30).

(A1) There exists a constant l such that for all $i = 1, 2, \dots, N$

$$|a(x, e_i)|^2 + |b(x, e_i)|^2 + \int_{\mathbb{R}^d} |g_1(x, e_i, u)|^2 m_1(du) \leq l(1 + |x|^2).$$

(A2) For any $r > 0$ one can specify constant l_r such that for all $i = 1, 2, \dots, N$

$$\begin{aligned} &|a(x, e_i) - a(y, e_i)|^2 + |b(x, e_i) - b(y, e_i)|^2 \\ &+ \int_{\mathbb{R}^d} |g_1(x, e_i, u) - g_1(y, e_i, u)|^2 m_1(du) \leq l_r |x - y|^2 \end{aligned}$$

for $|x| \leq r, |y| \leq r$.

(A3) Function c satisfies (2.31), (2.32), and for $i, j = 1, 2, \dots, N$, $\lambda(e_i, e_j, \cdot)$ are bounded and measurable, $\lambda(e_i, e_j, \cdot) \geq 0$.

(A4) Function g_2 satisfies (2.31), (2.33), and for all $t > 0, i, j = 1, \dots, N$

$$\int_0^t \int_{\mathbb{R}^d} |\phi(x, e_i, e_j, \underline{u})| p_2(ds, du) < \infty, \quad P\text{-a.s.}$$

THEOREM 2.6 Assume (A1)–(A4). Let p_1, p_2, W, X_0 and θ_0 be independent. Then SDE (2.29)–(2.30) has a unique strong solution which is a semimartingale.

PROOF See Theorem 4.1 in [15]. ■

In order to explicitly show the hybrid jump behavior as a strong solution to an SDE, Blom [2] has developed an approach to prove that solution of (2.29)–(2.30) is indistinguishable from the solution of the following set of Equations:

$$d\theta_t = \sum_{i=1}^N (e_i - \theta_{t-}) p_2 \left(dt, (\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-})) \times \mathbb{R}^{d-1} \right), \quad (2.34)$$

$$\begin{aligned} dX_t &= a(X_t, \theta_t) dt + b(X_t, \theta_t) dW_t + \int_{\mathbb{R}^d} g_1(X_{t-}, \theta_{t-}, u) q_1(dt, du) \\ &+ \int_{\mathbb{R}^d} \phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u}) p_2(dt, (0, \Sigma_N(X_{t-}, \theta_{t-})) \times d\underline{u}). \end{aligned} \quad (2.35)$$

THEOREM 2.7 Assume (A1)–(A4). Let p_1, p_2, W, X_0 and θ_0 be independent. Then SDE (2.34)–(2.35) has a unique strong solution which is a semimartingale.

PROOF The proof consists of showing that the solution of (2.34)–(2.35) is indistinguishable from the solution of (2.29)–(2.30). Subsequently Theorem 2.7 is the consequence of Theorem 2.6.

Indeed, rewriting of (2.34) yields (2.30):

$$\begin{aligned}
d\theta_t &= \sum_{i=1}^N (e_i - \theta_{t-}) p_2 \left(dt, (\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-})) \times \mathbb{R}^{d-1} \right) \\
&= \int_{\mathbb{R}^d} \sum_{i=1}^N (e_i - \theta_{t-}) I_{(\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-}))}(\underline{u}_1) p_2(dt, d\underline{u}_1 \times d\underline{u}) \\
&= \int_{\mathbb{R}^d} c(X_{t-}, \theta_{t-}, \underline{u}) p_2(dt, d\underline{u}).
\end{aligned}$$

Next, since the first three right hand terms of (2.35) and (2.29) are equal, it remains to show that the fourth right hand term in (2.35) yields the fourth right hand term in (2.29) up to indistinguishability:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u}) p_2(dt, (0, \Sigma_N(X_{t-}, \theta_{t-})) \times d\underline{u}) \\
&= \int_{(0, \infty)} \int_{\mathbb{R}^{d-1}} \phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u}) I_{(0, \Sigma_N(X_{t-}, \theta_{t-}))}(\underline{u}_1) p_2(dt, d\underline{u}_1 \times d\underline{u}) \\
&= \int_{(0, \infty)} \int_{\mathbb{R}^{d-1}} \phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u}) \times \\
&\quad \times \sum_{i=1}^N I_{(\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-}))}(\underline{u}_1) p_2(dt, d\underline{u}_1 \times d\underline{u}) \\
&= \int_{(0, \infty)} \int_{\mathbb{R}^{d-1}} \sum_{i=1}^N [\phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u}) \times \\
&\quad \times I_{(\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-}))}(\underline{u}_1)] p_2(dt, d\underline{u}_1 \times d\underline{u}) \\
&= \int_{(0, \infty)} \int_{\mathbb{R}^{d-1}} \sum_{i=1}^N [\phi(X_{t-}, \theta_{t-}, \theta_{t-} + \Delta\theta_t, \underline{u}) \times \\
&\quad \times I_{(\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-}))}(\underline{u}_1)] p_2(dt, d\underline{u}_1 \times d\underline{u}) \\
&= \int_{(0, \infty)} \int_{\mathbb{R}^{d-1}} \sum_{i=1}^N [\phi(X_{t-}, \theta_{t-}, \theta_{t-} + (e_i - \theta_{t-}), \underline{u}) \times \\
&\quad \times I_{(\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-}))}(\underline{u}_1)] p_2(dt, d\underline{u}_1 \times d\underline{u}) \\
&= \int_{(0, \infty)} \int_{\mathbb{R}^{d-1}} \sum_{i=1}^N [\phi(X_{t-}, \theta_{t-}, e_i, \underline{u}) \times \\
&\quad \times I_{(\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-}))}(\underline{u}_1)] p_2(dt, d\underline{u}_1 \times d\underline{u}) \\
&= \int_{\mathbb{R}^d} g_2(X_{t-}, \theta_{t-}, \underline{u}) p_2(dt, d\underline{u}).
\end{aligned}$$

This completes the proof. ■

REMARK 2.6 We notice the interesting aspect that the presence of θ_t in ϕ (Equation (2.35)) explicitly shows that jump of $\{X_t\}$ depends on the switch

from θ_{t-} to θ_t , i.e., it is a hybrid jump. \blacksquare

2.5 Instantaneous Hybrid Jumps at a Boundary

Up to now we have considered $\mathbb{R}^n \times \mathbb{M}$ -valued processes the jumps and switches of which are driven by Poisson random measure. In this section we will consider $\mathbb{R}^n \times \mathbb{M}$ -valued processes which also have instantaneous jumps and switches when hitting boundaries of some given sets. In order to simplify the analysis we assume that the purely discontinuous martingale term is equal to zero (i.e., we take $g_1 \equiv 0$).

First we define a particular sequence of processes. Suppose for each $e_i \in \mathbb{M}$, $i = 1, \dots, N$ there is an open connected set $E^i \subset \mathbb{R}^n$, with boundary ∂E^i . Let

$$E = \{x \mid x \in E^i, \text{ for some } i = 1, \dots, N\} = \bigcup_{i=1}^N E^i,$$

$$\partial E = \{x \mid x \in \partial E^i, \text{ for some } i = 1, \dots, N\} = \bigcup_{i=1}^N \partial E^i.$$

The interior of the set E is the jump “destination” set. Suppose that the function g_2 , defined by (2.33), in addition to requirement (A4) has the following property:

(B1) $(x + \phi(x, e_i, \underline{u})) \in E^i$ for each $x \in E^i$, $\underline{u} \in \mathbb{R}^{d-1}$, $i = 1, \dots, N$.

Similarly as in [3, pp. 38–39], we consider an increasing sequence of stopping times τ_n^E and a sequence of jump-diffusions $\{X_t^n; t \geq \tau_{n-1}^E\}$, $n = 1, 2, \dots$, governed by the following SDE (in integral form):

$$X_t^n = X_{\tau_{n-1}^E}^n + \int_{\tau_{n-1}^E}^t a(X_s^n, \theta_s^n) ds + \int_{\tau_{n-1}^E}^t b(X_s^n, \theta_s^n) dW_s \quad (2.36)$$

$$+ \int_{\tau_{n-1}^E}^t \int_{\mathbb{R}^d} g_2(X_{s-}^n, \theta_{s-}^n, u) p_2(ds, du),$$

$$\theta_t^n = \theta_{\tau_{n-1}^E}^n + \int_{\tau_{n-1}^E}^t \int_{\mathbb{R}^d} c(X_{s-}^n, \theta_{s-}^n, u) p_2(ds, du), \quad (2.37)$$

$$X_{\tau_n^E}^{n+1} = g^x(X_{\tau_n^E}^n, \theta_{\tau_n^E}^n, \beta_{\tau_n^E}^n), \quad (2.38)$$

$$\theta_{\tau_n^E}^{n+1} = g^\theta(X_{\tau_n^E}^n, \theta_{\tau_n^E}^n, \beta_{\tau_n^E}^n). \quad (2.39)$$

More specifically, the stopping times are defined as follows.

$$\tau_k^E \triangleq \inf\{t > \tau_{k-1}^E : X_t^k \in \partial E\}, \quad (2.40)$$

$$\tau_0^E \triangleq 0 \quad (2.41)$$

$k = 1, 2, \dots, N$, i.e., $\tau_0^E < \tau_1^E < \dots < \tau_k^E < \dots$ a.s.,

$$g^x : \partial E \times \mathbb{M} \times V \rightarrow \mathbb{R}^n, \quad (2.42)$$

$$g^\theta : \partial E \times \mathbb{M} \times V \rightarrow \mathbb{M}, \quad (2.43)$$

and $\{\beta_t, t \in [0, \infty)\}$ is the sequence of V -valued (one may take $V = \mathbb{R}^d$) i.i.d. random variables distributed according to some given distribution. The initial values X_0^1 and θ_0^1 are some prescribed random variables.

REMARK 2.7 Assumption (B1) ensures that the sequence of stopping times (2.40) is well defined and the boundary ∂E can be hit only by the continuous part

$$X_t^{c,n} = X_{\tau_{n-1}^E}^n + \int_{\tau_{n-1}^E}^t a(X_s^n, \theta_s^n) ds + \int_{\tau_{n-1}^E}^t b(X_s^n, \theta_s^n) dW_s \quad (2.44)$$

of the processes $\{X_t^n\}$, $n = 1, 2, \dots$, between the jumps and/or switching times generated by Poisson random measure p_2 . ■

In order to prove existence and uniqueness, we define the process $\{X_t, \theta_t\}$ as follows.

$$\begin{cases} X_t(\omega) = \sum_{n=1}^{\infty} X_t^n(\omega) I_{[\tau_{n-1}^E(\omega), \tau_n^E(\omega))}(t) \\ \theta_t(\omega) = \sum_{n=1}^{\infty} \theta_t^n(\omega) I_{[\tau_{n-1}^E(\omega), \tau_n^E(\omega))}(t) \end{cases} \quad (2.45)$$

provided there exist solutions $\{X_t^n, \theta_t^n\}$ of SDE (2.36)–(2.39). On the open set E , process $\{X_t, \theta_t\}$ (provided it exists) evolves according to SDE (2.29)–(2.30) or (2.34)–(2.35). At times τ_k^E there is a jump and/or switching determined by the mappings g^x and g^θ correspondingly, i.e., $X_{\tau_k^E} \neq X_{\tau_k^E-}$ and/or $\theta_{\tau_k^E} \neq \theta_{\tau_k^E-}$.

To ensure the existence of a strong unique solution of (2.45) we need assumption (B1) and the following:

(B2) $d(\partial E, g^x(\partial E, \mathbb{M}, V)) > 0$, i.e., $\{X_t\}$ may jump only inside of open set E .

(B3) Process (2.45) hits the boundary ∂E a.s. finitely many times on any finite time interval.

THEOREM 2.8 Assume (A1)–(A4) and (B1)–(B3). Let W , p_2 , $\{\beta_t, t \in [0, \infty)\}$, X_0 and θ_0 be independent. Then process (2.45) exists for every $t \in \mathbb{R}_+$, it is strongly unique and it is a semimartingale.

PROOF See Theorem 5.2 in [15]. ■

2.6 Related SDE Models on Hybrid State Spaces

In this section we compare stochastic hybrid models developed by Blom [2], Blom *et al.* [3], and Ghosh and Bagchi [9] with the models presented in Sections 2.4 and 2.5. We will use the same notations and definitions of coefficients as in Sections 2.4 and 2.5. Table 2.2 lists the models we are dealing within this section.

Table 2.2: List of models and their main features.

	θ	$X1$	$X2$	$\theta \& X2$	B
HB1 [2]	✓	-	✓	✓	-
HB2 [3]	✓	-	✓	✓	✓
GB1 [9]	✓	-	✓	✓	-
GB2 [9]	✓	-	-	-	✓
KB1 [15]	✓	✓	✓	✓	-
KB2 [15]	✓	-	✓	✓	✓

The conventions used in Table 2.2 have the following meaning:

HB1 refers to switching hybrid-jump diffusion of Blom [2];

HB2 refers to switching hybrid-jump diffusion with hybrid jumps at the boundary of Blom *et al.* [3];

GB1 refers to switching jump diffusion of Ghosh and Bagchi [9];

GB2 refers to switching diffusion with hybrid jumps at the boundary of Ghosh and Bagchi [9];

KB1 refers to switching hybrid-jump diffusion developed in Section 2.4;

KB2 refers to switching hybrid-jump diffusion with hybrid jumps at the boundary developed in Section 2.5.

θ stands for independent random switching of θ_t ;

$X1$ stands for independent random jump of X_t generated by compensated Poisson random measure;

$X2$ stands for independent random jump of X_t generated by Poisson random measure;

$\theta \& X2$ stands for simultaneous jump of X_t and θ_t generated by Poisson random measure;

B stands for simultaneous jump of X_t and θ_t at the boundary.

Stochastic hybrid model HB1 [2] forms a subset of KB1. The difference is that HB1 assumes a zero martingale measure q_1 in (2.29) or (2.34). Thanks to [16], Blom [2] also develops a verifiable version of condition (A4):

(A4') For any $k \in \mathbb{N}$, there exists a constant N_k such that for each $i, j \in \{1, 2, \dots, N\}$

$$\sup_{|x| \leq k} \int_{\mathbb{R}^{d-1}} |\phi(x, e_i, e_j, \underline{u})| \bar{\mu}(d\underline{u}) \leq N_k.$$

Stochastic hybrid model HB2 [3] equals KB2; [3] also develops the verifiable version (A4') of (A4). In order to explain the relation with GB1 and GB2 we first specify these stochastic hybrid models developed in [9].

2.6.1 Stochastic Hybrid Model GB1 of Ghosh and Bagchi

Now, let us consider the model GB1 of Ghosh and Bagchi [9].

The evolution of $\mathbb{R}^n \times \mathbb{M}$ -valued Markov process $\{X_t, \theta_t\}$ is governed by the following equations:

$$dX_t = a(X_t, \theta_t)dt + b(X_t, \theta_t)dW_t + \int_{\mathbb{R}} g(X_{t-}, \theta_{t-}, u)p(dt, du), \quad (2.46)$$

$$d\theta_t = \int_{\mathbb{R}} h(X_{t-}, \theta_{t-}, u)p(dt, du). \quad (2.47)$$

Here:

- (i) for $t = 0$, X_0 is a prescribed \mathbb{R}^n -valued random variable.
- (ii) for $t = 0$, θ_0 is a prescribed \mathbb{M} -valued random variable.
- (iii) W is an n -dimensional standard Wiener process.
- (iv) $p(dt, du)$ is a Poisson random measure with intensity $dt \times m(du)$, where m is the Lebesgue measure on \mathbb{R} . p is assumed to be independent of W .

The coefficients are defined as:

$$\begin{aligned} a &: \mathbb{R}^n \times \mathbb{M} \rightarrow \mathbb{R}^n \\ b &: \mathbb{R}^n \times \mathbb{M} \rightarrow \mathbb{R}^{n \times n} \\ g &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R} \rightarrow \mathbb{R}^n \\ h &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R} \rightarrow \mathbb{R}^N. \end{aligned}$$

Function h is defined as:

$$h(x, e_i, u) = \begin{cases} e_j - e_i & \text{if } u \in \Delta_{ij}(x) \\ 0 & \text{otherwise,} \end{cases} \quad (2.48)$$

where for $i, j \in \{1, \dots, N\}$, $i \neq j$, $x \in \mathbb{R}^n$, $\Delta_{ij}(x)$ are the intervals of the real line defined as:

$$\begin{aligned}\Delta_{12}(x) &= [0, \lambda_{12}(x)] \\ \Delta_{13}(x) &= [\lambda_{12}(x), \lambda_{12}(x) + \lambda_{13}(x)] \\ &\vdots \\ \Delta_{1N}(x) &= \left[\sum_{j=2}^{N-1} \lambda_{1j}(x), \sum_{j=2}^N \lambda_{1j}(x) \right) \\ \Delta_{21}(x) &= \left[\sum_{j=2}^N \lambda_{1j}(x), \sum_{j=2}^N \lambda_{1j}(x) + \lambda_{21}(x) \right)\end{aligned}$$

and so on. In general,

$$\Delta_{ij}(x) = \left[\sum_{i'=1}^{i-1} \sum_{\substack{j'=1 \\ j' \neq i'}}^N \lambda_{i'j'}(x) + \sum_{\substack{j'=1 \\ j' \neq i}}^{j-1} \lambda_{ij'}(x), \sum_{i'=1}^{i-1} \sum_{\substack{j'=1 \\ j' \neq i'}}^N \lambda_{i'j'}(x) + \sum_{\substack{j'=1 \\ j' \neq i}}^j \lambda_{ij'}(x) \right).$$

For fixed x these are disjoint intervals, and the length of $\Delta_{ij}(x)$ is $\lambda_{ij}(x)$, $\lambda_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}$, $i, j = 1, \dots, N$, $i \neq j$.

Let K_1 be the support of $g(\cdot, \cdot, \cdot)$ and let U_1 be the projection of K_1 on \mathbb{R} . It is assumed that U_1 is bounded. Let K_2 denote the support of $h(\cdot, \cdot, \cdot)$ and U_2 the projection of K_2 on \mathbb{R} . By definition of c , U_2 is a bounded set. One can define function $g(\cdot, \cdot, \cdot)$ so that the sets U_1 and U_2 form three nonempty sets: $U_1 \setminus U_2$, $U_1 \cap U_2$ and $U_2 \setminus U_1$ (see Figure 2.1). Then, we have the following:

(i) For $u \in U_1 \cap U_2$

$$\begin{cases} g(\cdot, \cdot, u) \neq 0 \\ h(\cdot, \cdot, u) \neq 0 \end{cases}$$

i.e., simultaneous jumps of X_t and switches of θ_t are possible.

(ii) For $u \in U_2 \setminus U_1$

$$\begin{cases} g(\cdot, \cdot, u) = 0 \\ h(\cdot, \cdot, u) \neq 0 \end{cases}$$

i.e., only random switches of θ_t are possible.

(iii) For $u \in U_1 \setminus U_2$

$$\begin{cases} g(\cdot, \cdot, u) \neq 0 \\ h(\cdot, \cdot, u) = 0 \end{cases}$$

i.e., only random jumps of X_t are possible.

Ghosh and Bagchi [9] proved that under the following conditions there exists an a.s. unique strong solution of SDE (2.46)–(2.47).

(D1) For each $e_i \in \mathbb{M}$, $i = 1, \dots, N$, $a(\cdot, e_i)$ and $b(\cdot, e_i)$ are bounded and Lipschitz continuous.

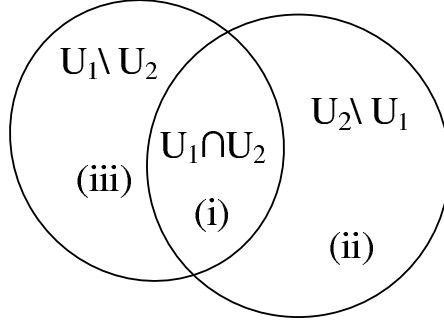


FIGURE 2.1: $U_1 \cup U_2$ is the projection of set $K_1 \cup K_2$ on \mathbb{R} .

- (D2) For all $i, j \in \{1, \dots, N\}$, $i \neq j$, functions $\lambda_{ij}(\cdot)$ are bounded and measurable, $\lambda_{ij}(\cdot) \geq 0$ for $i \neq j$ and $\sum_{j=1}^N \lambda_{ij}(\cdot) = 0$ for any $i \in \{1, \dots, N\}$.
- (D3) U_1 , the projection of support of $g(\cdot, \cdot, \cdot)$ on \mathbb{R} , is bounded.

2.6.2 Stochastic Hybrid Model GB2 of Ghosh and Bagchi

Next, we present the GB2 model of Ghosh and Bagchi [9]. The state of the system at time t , denoted by (X_t, θ_t) , takes values in $\bigcup_{n=1}^{\infty} (S_n \times \mathbb{M}_n)$, where $\mathbb{M}_n = \{e_1, e_2, \dots, e_{N_n}\}$ and $S_n \subset \mathbb{R}^{d_n}$. Between the jumps of X_t the state equations are of the form

$$dX_t = a^n(X_t, \theta_t)dt + b^n(X_t, \theta_t)dW_t^n, \quad (2.49)$$

$$d\theta_t = \int_{\mathbb{R}} h^n(X_{t-}, \theta_{t-}, u)p(dt, du), \quad (2.50)$$

where for each $n \in \mathbb{N}$

$$a^n : S_n \times \mathbb{M}_n \rightarrow \mathbb{R}^{d_n}$$

$$b^n : S_n \times \mathbb{M}_n \rightarrow \mathbb{R}^{d_n \times d_n}$$

$$h^n : S_n \times \mathbb{M}_n \times \mathbb{R} \rightarrow \mathbb{R}^{N_n}.$$

Function h^n is defined in a similar way as (2.48) with rates $\lambda_{ij}^n : S_n \rightarrow \mathbb{R}$, $\lambda_{ij}^n \geq 0$ for $i \neq j$, and $\sum_{j=1}^{N_n} \lambda_{ij}^n(\cdot) = 0$ for any $i \in \{1, \dots, N_n\}$. W^n is a standard d_n -dimensional Wiener process, and p is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with the intensity $dt \times m(du)$ as in the previous section.

For each $n \in \mathbb{N}$, let $A_n \subset S_n, D_n \subset S_n$. The set A_n is the set of instantaneous jump, whereas D_n is the destination set. It is assumed that for each $n \in \mathbb{N}$, A_n and D_n are closed sets, $A_n \cap D_n = \emptyset$ and $\inf_n d(A_n, D_n) > 0$, where $d(\cdot, \cdot)$ denotes the distance

between two sets. If at some random time X_t hits A_n , then it executes an instantaneous jump. The destination of (X_t, θ_t) at this juncture is determined by a map

$$g_n : A_n \times \mathbb{M}_n \rightarrow \cup_{m \in \mathbb{N}} (D_m \times \mathbb{M}_m).$$

After reaching the destination, the process $\{X_t, \theta_t\}$ follows the same evolutionary mechanism over and over again.

Let $\{\eta_t\}$ be an \mathbb{N} valued process defined by

$$\eta_t = n \text{ if } (X_t, \theta_t) \in S_n \times \mathbb{M}_n. \quad (2.51)$$

The $\{\eta_t\}$ is a piecewise constant process that changes from n to m when (X_t, θ_t) jumps from the regime $S_n \times \mathbb{M}_n$ to the regime $S_m \times \mathbb{M}_m$. Thus η_t is an indicator of a regime and a change in η_t means a switching in the regimes in which $\{X_t, \theta_t\}$ evolves.

Let

$$\begin{aligned} \tilde{S} &= \{(x, e_i, n) | x \in S_n, e_i \in \mathbb{M}_n\}, \\ \tilde{A} &= \{(x, e_i, n) | x \in A_n, e_i \in \mathbb{M}_n\}, \\ \tilde{D} &= \{(x, e_i, n) | x \in D_n, e_i \in \mathbb{M}_n\}. \end{aligned}$$

Then $\{X_t, \theta_t, \eta_t\}$ is an \tilde{S} -valued process, the set \tilde{A} is the set where jumps occur and \tilde{D} is the destination set for this process. The sets $\cup_n (S_n \times \mathbb{M}_n)$, $\cup_n (A_n \times \mathbb{M}_n)$, and $\cup_n (D_n \times \mathbb{M}_n)$ can be embedded in \tilde{S} , \tilde{A} , and \tilde{D} respectively.

Let d^0 denote the injection map of $\cup_n (D_n \times \mathbb{M}_n)$ into \tilde{D} . Define the maps \tilde{g}_1 , \tilde{g}_2 , and \tilde{h} as follows:

$$\begin{aligned} \tilde{g}_i &: \tilde{A} \rightarrow \tilde{D}, i = 1, 2, \\ \tilde{h} &: \tilde{A} \rightarrow \mathbb{N}, \end{aligned}$$

such that $\tilde{g}_1(x, e_i, n)$, $\tilde{g}_2(x, e_i, n)$ and $\tilde{h}(x, e_i, n)$ are the first, second and third component in $d^0(g_n(x, e_i))$ respectively. Let τ_{m+1} be the stopping time defined by

$$\tau_{m+1} = \inf\{t > \tau_m | X_{t-}, \theta_{t-}, \eta_{t-} \in \tilde{A}\}.$$

Now the equations for $\{X_t, \theta_t, \eta_t\}$ can be written as follows:

$$\begin{aligned} dX_t &= \left(a(X_t, \theta_t, \eta_t) + \sum_{m=0}^{\infty} [\tilde{g}_1(X_{\tau_m-}, \theta_{\tau_m-}, \eta_{\tau_m-}) - X_{\tau_m-}] \delta(t - \tau_m) \right) dt \\ &\quad + b(X_t, \theta_t, \eta_t) dW_t^{\eta_t}, \end{aligned} \quad (2.52)$$

$$\begin{aligned} d\theta_t &= \int_{\mathbb{R}} h(X_{t-}, \theta_{t-}, \eta_{t-}, u) p(dt, du) \\ &\quad + \sum_{m=0}^{\infty} [\tilde{g}_2(X_{\tau_m-}, \theta_{\tau_m-}, \eta_{\tau_m-}) - \theta_{\tau_m-}] \delta(t - \tau_m) dt, \end{aligned} \quad (2.53)$$

$$d\eta_t = \sum_{m=0}^{\infty} [\tilde{h}(X_{\tau_m-}, \theta_{\tau_m-}, \eta_{\tau_m-}) - \eta_{\tau_m-}] I_{\{\tau_m \leq t\}}, \quad (2.54)$$

where δ is the Dirac measure and $a(x, e_i, n) = a^n(x, e_i)$, $b(x, e_i, n) = b^n(x, e_i)$, and $h(x, e_i, n, u) = h^n(x, e_i, u)$.

To ensure the existence of an a.s. unique strong solution of SDE (2.52)–(2.54), Ghosh and Bagchi [9] adopted the following assumptions:

- (E1) For each $n \in \mathbb{N}$ and $e_i \in \mathbb{M}_i$, $a^n(\cdot, e_i)$ and $b^n(\cdot, e_i)$ are bounded and Lipschitz continuous.
- (E2) For each $n \in \mathbb{N}$, $i, j = 1, \dots, M_n$, $i \neq j$, functions $\lambda_{ij}^n(\cdot)$ are bounded and measurable, $\lambda_{ij}^n(\cdot) \geq 0$ for $i \neq j$ and $\sum_{j=1}^N \lambda_{ij}^n(\cdot) = 0$ for any $i \in \{1, \dots, N\}$.
- (E3) The maps g_n , $n \in \mathbb{N}$, are bounded and uniformly continuous.
- (E4) $\inf_n d(A_n, D_n) > 0$.

2.6.3 Hierarchy Between Stochastic Hybrid Models

In this subsection we discuss the differences between the models and determine the hierarchy of these models. This hierarchy is organized on the basis of the behaviors of the processes, e.g., different types of jumps, and not on the assumptions applied to the models. We summarize this hierarchy of models in Figure 2.2.

First, let us compare GB1 and HB1 (=KB1 with $g_1 = 0$). Both models allow either independent or simultaneous jumps and switches of X_t and θ_t . However, there are some differences in assumptions imposed on the coefficients and in construction of the jump and switching coefficients. The first two terms (i.e., the drift and the diffusion term) in (2.29) and in (2.46) are identical. However, to assure the existence of a strong unique solution of SDE (2.46)–(2.47), Ghosh and Bagchi [9] assume that the drift and the diffusion coefficients are bounded, i.e., condition (D1). To prove the similar result for SDE (2.29)–(2.30) more general growth condition (A1) is adopted. The construction of the “switching” terms (2.30) and (2.47) is almost identical with some minor differences in defining the “rate” intervals. The conditions on the “rate” functions $\lambda(e_i, e_j, \cdot)$ and $\lambda_{ij}(\cdot)$ are the same, i.e., these functions are assumed to be bounded and measurable for all $i, j = 1, \dots, N$, i.e., conditions (A3) and (D2).

There is a substantial difference in the construction of the g_2 jump part of X_t in the HB1/KB1 and GB1 models. In GB1 the jumps of X_t are described by a stochastic integral of function g with respect to a Poisson random measure $p(dt, du)$ with intensity $dt \times m(du)$, where m is the Lebesgue measure on $U = \mathbb{R}$. In order to satisfy the existence and uniqueness of solution, U_1 , the projection of support of function g on $U = \mathbb{R}$, must be bounded, i.e., condition (D3). In HB1/KB1 the g_2 jumps of X_t are also defined by a stochastic integral driven by Poisson random measure $p_2(dt, du)$ but with intensity $dt \times m(du_1) \times \bar{\mu}(\underline{u})$, where m is the Lebesgue measure on $U_1 = \mathbb{R}$ and $\bar{\mu}$ is a probability measure on $\underline{U} = \mathbb{R}^{d-1}$. The integrand function g_2 , which determines the jump size of X_t , compared to function g , has an extra argument

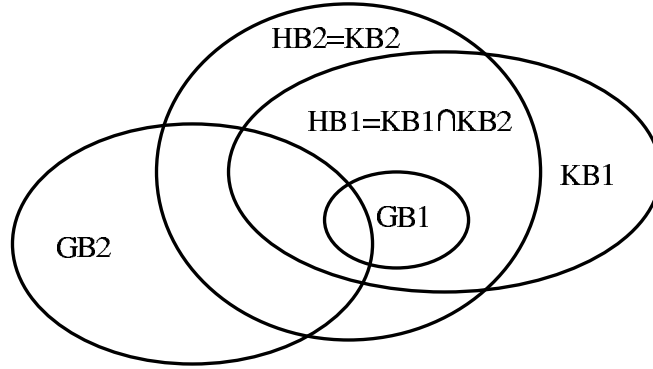


FIGURE 2.2: The hierarchy between stochastic hybrid models; the sets $HB2=KB2$ and $GB2$ fall within the set of Generalized Stochastic Hybrid Processes [4]. $KB1$ provides complementary modelling power in allowing processes that have infinite variation in jumps on a finite time interval.

$\underline{u} \in \underline{U} = \mathbb{R}^{d-1}$, and, since the intensity of p_2 with respect to \underline{u} is a probability measure $\bar{\mu}$ (which is always finite), the projection of support of g_2 on $\underline{U} = \mathbb{R}^{d-1}$ can be unbounded. This gives some extra freedom in modelling the jumps of X_t component. It is only required that function g_2 must satisfy condition (A4) or the verifiable (A4'). From this follows that model $HB1/KB1$ includes model $GB1$ as a special case ($GB1 \subset HB1 \subset KB1$).

Models $KB2$ and $GB2$ have some similarities. Let us see what are the main differences between SDE (2.36)–(2.39) and SDE (2.52)–(2.54). Solutions of SDE (2.52)–(2.54) are the $\bigcup_{n=1}^{\infty} (S_n \times \mathbb{M}_n)$ -valued switching diffusions with hybrid jumps at the boundary. Before hitting the boundary $\{X_t, \theta_t\}$ evolves as an $(S_n \times \mathbb{M}_n)$ -valued switching diffusion in some regime $\eta_t = n \in \mathbb{N}$. The drift and the diffusion coefficients and the mapping determining a new starting point of the process after the hitting the boundary can be different for every different regime $n \in \mathbb{N}$.

Solutions of SDE (2.36)–(2.39) are the $(\mathbb{R}^n \times \mathbb{M})$ -valued switching-jump diffusions with hybrid jumps at the boundary. The dimension of the state space and the coefficients of SDE are fixed. Hence, on this specific point, model $GB2$ is more general. However the jump term in $KB2$, see Equation (2.36), is more general than the jump term in $GB2$, see Equation (2.52).

Now let us have a look at conditions (E1)–(E4). Condition (E1) implies that our local conditions (A1) and (A2) for SDE (2.29)–(2.30) are definitely satisfied. Conditions (E2) and (E3) imply that conditions (A3) and (A4) for SDE (2.29)–(2.30) are satisfied. Condition (E4) implies that (B1) and (B2) adopted to SDE (2.36)–(2.39) are satisfied. It ensures that after the jump the process starts inside of some open set, but not on a boundary. Condition (B3) of SDE (2.36)–(2.39) is missing for $GB2$ [9].

In general $GB2$ is not a subclass of $KB2$ (or $HB2$) since in $GB2$ the state of the system (X_t, θ_t) takes values in $\bigcup_{k=1}^{\infty} (S_k \times \mathbb{M}_k)$, where $\mathbb{M}_k = \{e_1, e_2, \dots, e_{N_k}\}$ and

$S_k \subset \mathbb{R}^{d_k}$ may be different for different k 's. If $(S_k \times \mathbb{M}_k) = (\mathbb{R}^n \times \mathbb{M})$ for all $k \in \mathbb{N}$ then obviously GB2 \subset KB2 (=HB2).

2.7 Markov and Strong Markov Properties

In this section we prove Markov and strong Markov properties for model HB2=KB2 (Section 2.5).

Assume we are given the following objects:

- A measurable space (S, \mathcal{S}) .
- Another measurable space (Ω, \mathcal{G}) and a family of σ -algebras $\{\mathcal{G}_t^s, 0 \leq s \leq t \leq \infty\}$, such that $\mathcal{G}_t^s \subset \mathcal{G}_v^u \subset \mathcal{G}$ provided $0 \leq u \leq s \leq t \leq v$; \mathcal{G}_t^s denotes a σ -algebra of events on time interval $[s, t]$; we write \mathcal{G}_t in place of \mathcal{G}_t^0 and \mathcal{G}^s in place of \mathcal{G}_∞^s .
- A probability measure $P_{s,x}$ for each pair $(s, x) \in [0, \infty) \times S$ on \mathcal{G}^s .
- A function (stochastic process) $\xi_t(\omega) = \xi(t, \omega)$ defined on $[0, \infty) \times \Omega$ with values in S .

The system consisting of these four objects will be denoted by $\{\xi_t, \mathcal{G}_t^s, P_{s,x}\}$ [10].

DEFINITION 2.7 *A system of objects $\{\xi_t, \mathcal{G}_t^s, P_{s,x}\}$ is called a Markov process provided:*

- (i) for each $t \in [0, \infty)$ $\xi_t(\omega)$ is measurable mapping of (Ω, \mathcal{G}) into (S, \mathcal{S}) ;
- (ii) for arbitrary fixed s, t and B ($0 \leq s \leq t, B \in \mathcal{S}$) the function $P(s, x, t, B) = P_{s,x}(\xi_t \in B)$ is \mathcal{S} -measurable with respect to x ;
- (iii) $P_{s,x}(\xi_s = x) = 1$ for all $s \geq 0$ and $x \in S$;
- (iv) $P_{s,x}(\xi_u \in B | \mathcal{G}_t^s) = P_{t, \xi_t}(\xi_u \in B)$ for all $s, t, u, 0 \leq s \leq t \leq u < \infty, x \in S$ and $B \in \mathcal{S}$.

The measure $P_{s,x}$ should be considered as a probability law which determines the probabilistic properties of the process $\xi_t(\omega)$ given that it starts at point x at the time s . Condition (iv) in Definition 2.7 expresses the Markov property of the processes. Let $\mathbb{E}_{s,x}$ denote the expectation with respect to measure $P_{s,x}$. For \mathcal{G}^s -measurable random variable $\xi(\omega)$

$$\mathbb{E}_{s,x}[\xi(\omega)] = \int \xi(\omega) P_{s,x}(d\omega).$$

It is not difficult to show that the Markov property (iv) in Definition 2.7 can be rewritten in terms of expectations as follows:

$$\mathbb{E}_{s,x}[f(\xi_u) \mid \mathcal{G}_t^s] = \mathbb{E}_{t,\xi_t}[f(\xi_u)], \quad 0 \leq s \leq t \leq u < \infty,$$

where f is an arbitrary \mathcal{S} -measurable bounded function.

Next, let us show that process

$$\begin{cases} X_t(\omega) = \sum_{n=1}^{\infty} X_t^n(\omega) I_{[\tau_{n-1}^E(\omega), \tau_n^E(\omega))}(t) \\ \theta_t(\omega) = \sum_{n=1}^{\infty} \theta_t^n(\omega) I_{[\tau_{n-1}^E(\omega), \tau_n^E(\omega))}(t) \end{cases} \quad (2.55)$$

defined as a concatenation of solutions $\{X_t^n, \theta_t^n\}$ of the system of SDE (2.36)–(2.39) (see Sections 2.4 and 2.5) is Markov. We follow the approach used in [11]. Let $\xi_t^{s,\eta} = (X_t^{s,x}, \theta_t^{s,\theta})$ denote the process (2.55) on $[s, \infty)$ satisfying initial condition $\xi_s^{s,\eta} = \eta = (X_s^{s,x}, \theta_s^{s,\theta})$. Note that now $S = \mathbb{R}^n \times \mathbb{M}$ and $\mathcal{S} = \mathcal{B}_{\mathbb{R}^n \times \mathbb{M}}$ is the σ -algebra of Borel sets on $\mathbb{R}^n \times \mathbb{M}$. Assume that conditions of Theorem 2.8 are satisfied. Let \mathcal{F}_t^s , $s < t$ be the σ -algebras generated by $\{W_u - W_s, p_2([s, u], dz), \beta_u, u \in [s, t]\}$, $\mathcal{F}_t^0 = \mathcal{F}_t$, $\mathcal{F}_\infty^s = \mathcal{F}^s$. For $s \leq t$ the σ -algebras \mathcal{F}_s and \mathcal{F}^s are independent. Process $\xi_t^{s,\eta}$ is \mathcal{F}^s -measurable, hence, it is independent of σ -algebra \mathcal{F}_s . Let η_s be an arbitrary $\mathbb{R}^n \times \mathbb{M}$ -valued \mathcal{F}_s measurable random variable. Then ξ_t^{s,η_s} , $t \geq s$, is unique \mathcal{F}_t -measurable process on $[s, \infty)$ satisfying the initial condition $\xi_s^{s,\eta_s} = \eta_s$. Since for $u < s$ process $\xi_t^{u,y}$ is \mathcal{F}_t -measurable on $[s, \infty)$ with initial condition $\xi_s^{u,y}$ then the following equality holds

$$\xi_t^{u,y} = \xi_t^{s,\xi_s^{u,y}}, \quad u < s < t. \quad (2.56)$$

Let φ be a bounded measurable function on $\mathbb{R}^n \times \mathbb{M}$, let ζ_s be an arbitrary bounded \mathcal{F}_s -measurable quantity. The independence of \mathcal{F}_s and \mathcal{F}^s and the Fubini theorem imply that measure P on \mathcal{F}_∞ is a product of measures P_s and P^s , where P_s is a restriction of P on \mathcal{F}_s , where P^s is a restriction of P on \mathcal{F}^s , and

$$\mathbb{E}[\varphi(\xi_t^{u,y}) \zeta_s] = \mathbb{E}[\varphi(\xi_t^{s,\xi_s^{u,y}}) \zeta_s] = \mathbb{E}[\zeta_s (\mathbb{E}[\varphi(\xi_t^{s,x})])_{x=\xi_s^{u,y}}].$$

Since $\xi_s^{u,y}$ is \mathcal{F}_s -measurable then $\mathbb{E}[\varphi(\xi_t^{u,y}) \mid \mathcal{F}_s] = [\mathbb{E}[\varphi(\xi_t^{s,x})]]_{x=\xi_s^{u,y}}$. Let

$$P(s, x, t, B) = P(\xi_t^{s,x} \in B), \quad B \in \mathcal{B}_{\mathbb{R}^n \times \mathbb{M}}, \quad (2.57)$$

here $\mathcal{B}_{\mathbb{R}^n \times \mathbb{M}}$ is the σ -algebra of Borel sets on $\mathbb{R}^n \times \mathbb{M}$. Then, by taking $\varphi = I_B$, we obtain

$$P(\xi_t^{u,y} \in B \mid \mathcal{F}_s) = P(s, \xi_s^{u,y}, t, B). \quad (2.58)$$

If ξ_t is an arbitrary process defined by (2.55), by the same reasoning with help of which equalities (2.56) and (2.58) have been obtained, one can show that $\xi_t = \xi_t^{s,\xi_s}$ for $s < t$ and that

$$P(\xi_t \in B \mid \mathcal{F}_s) = P(s, \xi_s, t, B).$$

Hence, the process defined by (2.55) is a Markov process with transition probability $P(s, x, t, B)$ defined by (2.58). To be precise, we have shown that the system of objects $\{(X_t, \theta_t), \mathcal{F}_t^s, P_{s, (x, \theta)}\}$, where $P_{s, (x, \theta)}((X_t, \theta_t) \in B) = P(s, (x, \theta), t, B) = P((X_t^{s, x}, \theta_t^{s, \theta}) \in B)$, $B \in \mathcal{B}_{\mathbb{R}^n \times \mathbb{M}}$, is a Markov process.

Next, we prove the Markov property

$$P_{s, x}(\xi_u \in B \mid \mathcal{G}_t^s) = P_{t, \xi_t}(\xi_u \in B), \quad s \leq t \leq u$$

remains valid also when a fixed time moment t is replaced by a stopping time.

Let $\{\xi_t(\omega), \mathcal{G}_t^s, P_{s, x}\}$ be a Markov process in the space (S, \mathcal{S}) . Let \mathcal{T} denote the σ -algebra of Borel sets on $[0, \infty)$.

DEFINITION 2.8 *A Markov process is called strong Markov if:*

- (i) *the transition probability $P(s, x, t, B)$ for a fixed B is a $\mathcal{T} \times \mathcal{S} \times \mathcal{T}$ -measurable function of (s, x, t) on the set $0 \leq s \leq t < \infty$, $x \in S$;*
- (ii) *it is progressively measurable;*
- (iii) *for any $s \geq 0$, $t \geq 0$, \mathcal{S} -measurable function $f(x)$ and arbitrary stopping time τ ,*

$$\mathbb{E}_{s, x}[f(\xi_{t+\tau}) \mid \mathcal{G}_\tau^s] = \mathbb{E}_{\tau, \xi_\tau}[f(\xi_{t+\tau})]. \quad (2.59)$$

REMARK 2.8 For Equation (2.59) to be satisfied, it is necessary that the random variable $g(\xi_\tau, \tau, t + \tau) = \mathbb{E}_{\tau, \xi_\tau}[f(\xi_{t+\tau})]$ be \mathcal{G}_τ^s -measurable. For this reason assumptions (i) and (ii) make part of the definition of the strong Markov property [10]. ■

Now we return to the process $\xi_t = (X_t, \theta_t)$ defined in Section 2.5. We have shown that it is a Markov process. The following proposition proves that it is a strong Markov process also.

PROPOSITION 2.1 *Assume (A1)–(A4) and (B1)–(B3). Let W , p_2 , μ^E , X_0 and θ_0 be independent. Let \mathcal{F}_t^s , $s < t$ be the σ -algebras generated by $\{W_u - W_s, p_2(dz, [s, u]), \beta_u, u \in [s, t]\}$. For any bounded Borel function $f: \mathbb{R}^n \times \mathbb{M} \rightarrow \mathbb{R}$ and any \mathcal{F}_t^s -stopping time τ*

$$\mathbb{E}_{s, x}[f(\xi_{t+\tau}) \mid \mathcal{F}_\tau^s] = \mathbb{E}_{\tau, \xi_\tau}[f(\xi_{t+\tau})].$$

PROOF Let $\{\sigma_k, k = 0, 1, \dots\}$ denote the ordered set of the stopping times $\{\tau_k^E, k = 1, 2, \dots\}$ and $\{\tau_k, k = 0, 1, \dots\}$. The latter set is the set of the stopping times generated by Poisson random measure p_2 . Then on each time

interval $[\sigma_{k-1}, \sigma_k)$, $k = 1, 2, \dots$ process ξ_t evolves as a diffusion starting at point $\xi_{\sigma_{k-1}}$ at the time σ_{k-1} . This means that on each time interval $[\sigma_{k-1}, \sigma_k)$ the strong Markov property holds. Let \mathcal{F}_τ^s be the σ -algebra generated by the \mathcal{F}_t^s -stopping time τ . The sets $\{\omega : \tau(\omega) \in [\sigma_{k-1}(\omega), \sigma_k(\omega))\}$, $k = 1, 2, \dots$ are \mathcal{F}_τ^s -measurable. Hence

$$\begin{aligned} \mathbb{E}_{s,x}[f(\xi_{t+\tau}) \mid \mathcal{F}_\tau^s] &= \sum_{k=0}^{\infty} I_{[\sigma_{k-1}, \sigma_k)}(\tau) \mathbb{E}_{s,x}[f(\xi_{t+\tau}) \mid \mathcal{F}_\tau^s] \\ &= \sum_{k=0}^{\infty} \mathbb{E}_{s,x}[I_{[\sigma_{k-1}, \sigma_k)}(\tau) f(\xi_{t+\tau}) \mid \mathcal{F}_\tau^s] \\ &= \sum_{k=0}^{\infty} \mathbb{E}_{\tau, \xi_\tau}[I_{[\sigma_{k-1}, \sigma_k)}(\tau) f(\xi_{t+\tau})] \\ &= \mathbb{E}_{\tau, \xi_\tau} \left[\sum_{k=0}^{\infty} I_{[\sigma_{k-1}, \sigma_k)}(\tau) f(\xi_{t+\tau}) \right] \\ &= \mathbb{E}_{\tau, \xi_\tau} [f(\xi_{t+\tau})]. \end{aligned}$$

This completes the proof. ■

2.8 Concluding Remarks

We have given an overview of stochastic hybrid processes as strongly unique solutions to stochastic differential equations on hybrid state space. These SDEs are driven by Brownian motion and Poisson random measure. Our overview has shown several new classes of stochastic hybrid processes each of which goes significantly beyond the well known class of jump-diffusions with Markov switching coefficients, whereas semimartingale and strong Markov properties have been shown to hold true. The main phenomena covered by these extensions are:

- Hybrid jumps, i.e., continuous valued jumps that happen simultaneously with a mode switch, and the size of which depends of the mode value prior and after the switch;
- Instantaneous jump reflection at the boundary, i.e., upon hitting a given measurable boundary of the Euclidean valued set, the continuous valued process component jumps instantaneously away from the boundary;
- The continuous valued process component may jump so frequently that it is no longer a process of finite variation;
- Feasible combinations of these phenomena within one SDE such that its solution still is a semimartingale strong Markov process.

For each of the extensions, our overview provides the specific conditions on the SDE under which there exist strongly unique semimartingale solutions. We also presented a novel approach to prove strong Markov property for general stochastic hybrid processes.

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