Nationaal Lucht- en Ruimtevaartlaboratorium

National Aerospace Laboratory NLR

Executive summary

A provably stable marching-on-in-time scheme based on quadratic spline basis functions

Problem area
Knowledge of the scattering characteristics of a platform is of great interest for military aircraft design and operation. The knowledge can for instance be applied to target identification. High range resolution profiles (HRRP) are considered to be the basis of reliable target identification algorithms. Next to experimental measurements, computational methods are increasingly being used as viable prediction tools. Prediction of HRRP with a frequency-domain method requires a significant number of radar frequencies to be simulated. Time-domain methods are expected to be more efficient.

Description of work
In order to extend the applicability of numerical tools, a promising method that is fully formulated in time-domain will be used to complement frequency-domain packages. The design of a robust time domain boundary integral equation (TDIE) method is a challenging task. In this report, available mathematical theory will be adapted and extended to obtain practical requirements for the choice of numerical parameters.

Results and conclusions
The two most popular discretization methods for TDIE methods are marching-on-in-time (MOT) and space-time Galerkin (STG). While MOT schemes are more efficient because of available accelerators, the STG scheme has been analysed thoroughly with mathematical theory. In particular, a stability proof has been derived for the STG scheme. In this report the stability proof for the STG scheme will first be summarized. Then, the STG and MOT schemes will be related with each other on a continous and discrete level. This allows for the translation of the STG stability proof into practical requirements for the choice of basis functions in the MOT scheme. The presented analysis is an important step towards robust numerical schemes that make this new computational method more reliable.

Applicability
Marching-on-in-time methods promise to extend the applicability of standard computational methods to radar absorbing materials and high range resolution profiles.

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A provably stable marching-on-in-time scheme based on quadratic spline basis functions

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Abstract

Computational methods are widely used in the engineering community for the analysis of electromagnetic scattering phenomena. To achieve the robustness required for industrial application the numerical scheme has to be provably stable. The existing stability proof of the space-time Galerkin scheme will be augmented such that it can be used for the more popular marching-on-in-time scheme. These extensions lead to a provably stable scheme that is easy to implement in existing marching methods.

Contents

1 Introduction

Electromagnetic scattering phenomena occur in many fields of engineering science, for example in the modification of the radar signature of an aircraft. Computational methods have widespread application in the analysis and design of scatterers. When the response has a nonlinear character, formulations in time-domain are preferred, for instance the Time Domain Integral Equation (TDIE) method.

Computational methods have to be robust in order to be applicable for engineering problems of industrial interest. In particular, stability of the underlying numerical scheme has to be guaranteed. This necessitates a mathematical basis of the discretization method that results in clear conditions for numerical stability.

Two of the discretization schemes for TDIE methods are the Marching-on-in-Time (MoT) scheme (Ref. 5) and the space-time Galerkin scheme (Ref. 1). The MoT scheme has been more popular because of its intuitive implementation and the inception of accelerators based on plane-wave and fast-Fourier techniques which have improved the efficiency to a great extent. On the other hand, late-time instabilities are still present in modern MoT schemes, depending on the application. Many methods have been introduced to remedy the instabilities, including filtering, accurate quadrature schemes and smooth basis functions (Ref. 5), (Ref. 2). Due to the lack of a thorough mathematical foundation for MoT schemes the present stabilization methods are mainly *ad hoc* and do not provide a *provably* stable discretization scheme. Hence, numerical stability has been experimentally demonstrated with extremely long experiments or an *a posteriori* eigenvalue analysis.

This paper aims to derive an *a priori* stability proof for the MoT scheme. We will build on the existing stability proof of the space-time Galerkin scheme for TDIE methods (Ref. 1). As opposed to the MoT scheme, a mathematical foundation for the space-time Galerkin scheme has been derived in (Ref. 3). This proves that the space-time Galerkin scheme is an unconditional stable scheme provided that test and basis functions are chosen carefully. Within this functional framework numerical stability is achieved regardless of the mesh sizes in space and time.

Although there is a resemblance between both numerical schemes, the available stability proof for space-time Galerkin schemes cannot be applied directly to MoT schemes. In this paper we will use the functional framework of stable space-time Galerkin schemes as a guideline for the design of temporal basis functions in the MoT scheme.

The main contribution of this paper is thus the extension of the existing stability proof for the space-time Galerkin scheme such that it can be applied to the more popular MoT scheme. With this novel approach of tackling the late-time instabilities we will show that the MoT scheme based on quadratic spline basis functions is provably stable.

2 Model equations

The TDIE method will be used to model the transient electromagnetic scattering of a PEC object. Scattered electromagnetic fields can be represented by the Stratton-Chu integral formulation in terms of the electric surface current density $\mathbf{J}$ on the scattering surface Γ. For $R = |\mathbf{r} - \mathbf{r}'|$, $\tau = t - \frac{R}{c}$ the retarded time, and $c = (\mu\epsilon)^{-\frac{1}{2}}$ the speed of light one has

$$\mathbf{E}(\mathbf{r}', t) = \int_\Gamma \Big(\mu \frac{\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r}, \tau)}{4\pi R} - \nabla' \frac{\int_{-\infty}^{\tau} \nabla \cdot \mathbf{J}(\mathbf{r}, \bar{t}) d\bar{t}}{\epsilon\, 4\pi R}\Big) d\mathbf{r}, \tag{1}$$

$$\dot{\mathbf{E}}(\mathbf{r}', t) = \int_\Gamma \Big(\mu \frac{\frac{\partial^2}{\partial t^2}\mathbf{J}(\mathbf{r}, \tau)}{4\pi R} - \nabla' \frac{\nabla \cdot \mathbf{J}(\mathbf{r}, \tau)}{\epsilon\, 4\pi R}\Big) d\mathbf{r} \tag{2}$$

the scattered field and its time derivative, respectively. The Electric Field Integral Equation (EFIE) is obtained by equating the tangential components of the scattered and incident field on the surface. The differentiated EFIE uses the differentiated fields and contains no integral in time.

3 Stability analysis

In this section the stability proof for MoT schemes will be derived. First, the conclusions of the existing stability proof for space-time Galerkin schemes will be summarized. Then we will present the two novelties of the proof, namely the relation between the original EFIE and the differentiated EFIE, and the relation between the two discretization schemes.

3.1 Space-time Galerkin scheme

Space-time Galerkin schemes reformulate the model equation into a variational problem. That is, define function spaces H_{sol} and H_{test} and search for a solution $\mathbf{J} \in H_{\text{sol}}$ of

$$b\left(\mathbf{g}, \mathbf{J}\right) = \left\langle \mathbf{g}, \mathbf{E}^{\text{inc}} \right\rangle_\sigma \quad \forall \mathbf{g} \in H_{\text{test}}. \tag{3}$$

The bilinear form b represents an inner product of the test function $\mathbf{g}$ and the original EFIE operator (1). A weighted inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle_\sigma = \int_0^\infty e^{-2\sigma t} \int_\Gamma \int_\Gamma \mathbf{p} \cdot \mathbf{q}\, ds ds'\, dt \tag{4}$$

is used with $\sigma > 0$. It has been proven that the variational problem (3) admits a unique solution for the spaces $H_{\text{sol}} = \mathcal{H}^{\frac{1}{2}}$ and

$$H_{\text{test}} = \mathcal{H}^{-\frac{3}{2}} \text{ with } \mathcal{H}^s = H^s_\sigma\big(\mathbb{R}_+, H^{-\frac{1}{2}}(\text{div}, \Gamma)\big) \tag{5}$$

specific Sobolev spaces as defined in (Ref. 3). Moreover, the solution is bounded by the incident field, i.e.,

$$||\mathbf{J}||_{H_{\text{sol}}} \leq C_\Gamma \frac{1}{\sigma} \left|\left|\mathbf{E}^{\text{inc}}\right|\right|_{H_{\text{inc}}}. \tag{6}$$

Since the norm is related to the electromagnetic energy, the boundedness of the continuous solution results in unconditional stability for a conforming Galerkin method (Ref. 1).

3.2 Relation between original EFIE and differentiated EFIE

The stability proof for space-time Galerkin schemes has been derived for the original EFIE. It cannot be applied readily to the differentiated EFIE which corresponds to a different bilinear form $\tilde{b}$ and the variational problem

$$\tilde{b}\,(\mathbf{h}, \mathbf{J}) = \left\langle \mathbf{h}, \tfrac{\partial}{\partial t}\mathbf{E}^{\text{inc}} \right\rangle_\sigma \quad \forall \mathbf{h} \in \tilde{H}_{\text{test}}. \tag{7}$$

To show this is equivalent with (3), use integration by parts. Because of the weighted inner product one obtains $b(\frac{\partial \mathbf{h}}{\partial t} - 2\sigma\mathbf{h}, \mathbf{J}) = \langle \frac{\partial \mathbf{h}}{\partial t} - 2\sigma\mathbf{h}, \mathbf{E}^{\text{inc}}\rangle_\sigma$. For

$$\mathbf{g} = \frac{\partial \mathbf{h}}{\partial t} - 2\sigma\mathbf{h} \tag{8}$$

it can be shown that for all $\mathbf{g} \in \mathcal{H}^{-\frac{3}{2}}$ there is an $\mathbf{h} \in \mathcal{H}^{-\frac{1}{2}}$ and vice versa. This shows the equivalence between the two variational problems (3) and (7). Therefore, boundedness result (6) holds for the spaces $H_{\text{sol}} = \mathcal{H}^{\frac{1}{2}}$ and $\tilde{H}_{\text{test}} = \mathcal{H}^{-\frac{1}{2}}$. So a conforming Galerkin method based on these Sobolev spaces is unconditionally stable for the differentiated EFIE.

3.3 Relation between space-time Galerkin and collocation

The MoT scheme uses *collocation* or *point matching* as discretization in time. This boils down to using a Dirac delta distribution as test function in the space-time Galerkin scheme. Unconditional stability of MoT schemes can therefore only be achieved if the Dirac delta is an element of $\tilde{H}_{\text{test}}$. Unfortunately, $\delta \notin \mathcal{H}^{-\frac{1}{2}}$ and stability of the MoT scheme cannot be proven for the differentiated EFIE directly. Still, this does not imply instability of the MoT scheme. In fact, we will show that for specific basis functions the MoT scheme is yet provably stable for the differentiated EFIE.

First, let us consider a stable choice of test and basis functions in the space-time Galerkin scheme, namely the *step* and *hat* functions, respectively. Because space-time Galerkin and collocation schemes have the same marching structure, the schemes can be compared at a discrete level. For MoT schemes with quadratic spline basis functions (Ref. 4) exactly the same discrete equations are obtained as for the stable choice in space-time Galerkin methods. Therefore, the use of quadratic spline basis functions results in a provably stable MoT scheme.

Now, let us use this extension of the stability proof to analyze the prevailing quadratic Langrange basis functions for MoT schemes (Ref. 2). The discrete equations are equivalent to the space-time Galerkin scheme using step and shifted hat functions as test and basis functions, respectively. As depicted in Fig. 1 the shifted hat function is discontinuous and is therefore no element of $H_{\text{sol}} = \mathcal{H}^{\frac{1}{2}}$. So, even with the extension of the stability proof, MoT schemes with quadratic Lagrange basis functions cannot be proven to be stable with the present analysis.

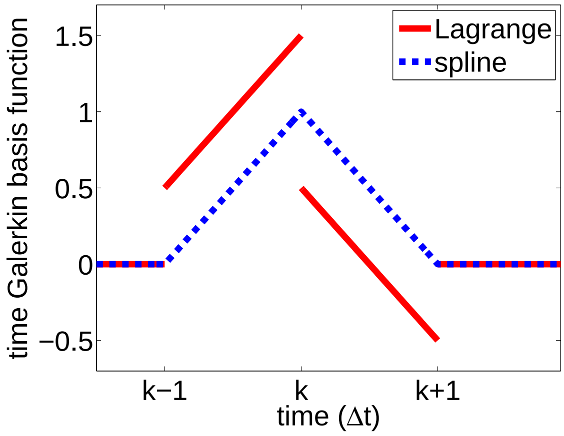

Fig. 1 Corresponding space-time Galerkin basis functions for the quadratic Lagrange and spline basis function.

3.4 Remark on stability

The variational problem corresponding to the original EFIE has been proven to be unconditionally stable for specific Sobolev spaces in (Ref. 3). This report uses this functional framework to obtain different Sobolev spaces for which the variational problem of the differentiated EFIE is stable. The MoT scheme with quadratic spline basis functions has been shown to fit within this functional framework. However, additional conditions are necessary to proof stability on a discrete level. These so-called *inf-sup conditions* can be elaborate and are outside the scope of this report.

4 Numerical validation

The stability of the MoT scheme with quadratic spline basis functions for the differentiated EFIE has been validated on a number of test cases and compared with the quadratic Lagrange basis functions. The common RWG functions are used for spatial discretization and the inner two spatial integrals have been computed analytically (Ref. 2) whereas 7 quadrature points are used for the outer two. As example, the surface current density depicted in Fig. 2 clearly shows a late-time instability for the quadratic Lagrange basis function whereas for the quadratic spline basis function the stability is confirmed.

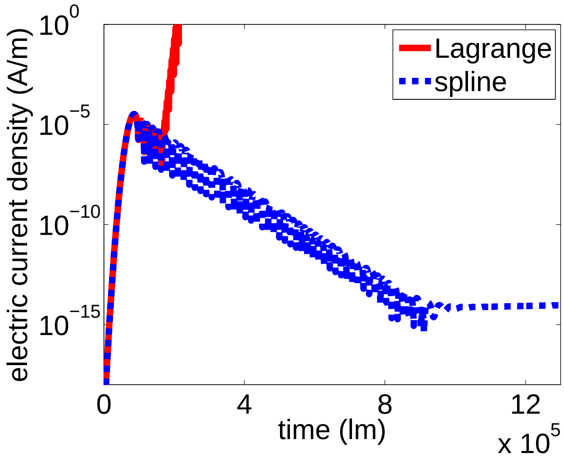

Fig. 2 The electric surface current density computed with the MoT scheme for the quadratic Lagrange and spline basis function.

5 Conclusion

The continuous variational problem of the differentiated EFIE has been proven to be unconditionally stable for specific Sobolev spaces. The quadratic spline basis functions used in the MoT scheme are elements of these stable function spaces.

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