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A system theoretical framework to study unilaterally constrained mechanical systems

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ABSTRACT
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CONTRIBUTION TO THE PROPOSED INVITED SESSION: HYBRID MECHANICAL SYSTEMS

A SYSTEM THEORETICAL FRAMEWORK TO STUDY UNILATERALLY CONSTRAINED MECHANICAL SYSTEMS

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Abstract

The aim of this paper is to study, in a system theoretical framework, the contact problem for mechanical systems subject to unilateral constraints. Contrary to what is usually done in studies on this subject, we first investigate what can be deduced already from the mathematical models of the unconstrained mechanical system and the constraint set. It is only after this analysis that we incorporate the physics of contact in our framework. This sequence of steps leads to important insights to the contact problem itself, and to useful theoretical findings with respect to modelling and control of impact.

1 Introduction

In this paper we study, in a system theoretical framework, the contact problem for mechanical systems subject to unilateral constraints. The contact problem arises when (a trajectory of) a mechanical system interacts with the boundary set of a unilateral constraint. In [7] we have studied the contact problem for linear dynamical systems. In this paper we will extend the results of [7] to a class of nonlinear systems: mechanical systems subject to unilateral constraints. A constrained robotic manipulator will illustrate our approach to the contact problem.

In hostile environments, such as space, a large number of operations will be executed autonomously by robotic manipulators [5]. During these operations there are many instances where a manipulator, specifically its end-effector, is, or comes, in contact with a fixed or moving external object even though operational control schemes may be aimed to avoid them [3, 6]. In general, collisions causes elastic oscillations and joint rotations, and in case of retrieval of large satellites possible attitude drift of the carrier spacecraft as well.

A primary goal of controller design for constrained manipulators is to ensure that impact forces remain within specified bounds (in order not to damage the structure), that bouncing of the manipulator is avoided as much as possible, and that post-impact dynamics interferes to a minimum with the task at hand. Here impact refers to the short period of time where the impulse due to collision enters the system, and post-impact refers to the period of time where the effects of impact, after the impact event itself has occurred, are notable in the system. Clearly, control of the (post-)impact is aimed also at shortening these periods. Interesting theoretical and experimental results have been presented recently, see e.g. [11, 13, 14, 16] and the references therein.

The remainder of this paper is organised as follows. In section 2 the problem formulation and some mathematical preliminaries are presented. In section 3 we investigate how trajectories of mechanical systems interact with the boundary of a unilateral constraint. In the present paper we will focus on a single unilateral constraint whose boundary set can be described (locally) by a linear equality. This will enable us to focus on the introduction of our framework. (The nonlinear case can be found in [8].) In section 4 we will introduce the collision maps that make the constraint set an invariant set for the constrained system. In section 5 we will incorporate the physics of contact into our framework, and give a general expression for the collision map for constrained mechanical systems. Some implications of our approach to the contact problem as far as control is concerned are discussed in section 6. Finally, the conclusions can be found in section 7.

2 Motivation and mathematical preliminaries

A basic dynamics equation for the manipulator reads [2, 10]:

$$M_b(q)\ddot{q} + N_b(q,\dot{q}) = \tau. \quad (1)$$

Here q denotes the generalized joint coordinates, $M_b(q)$ denotes the inertial matrix function, $N_b(q,\dot{q})$ is a vector function which characterizes the Coriolis, centrifugal and gravitational load of the manipulator, and τ is the vector of generalized inputs.

We assume the following relation between end-effector position y, and velocity $\dot{y}$, and joint values q, and joint angular values $\dot{q}$, respectively: $y = H(q)$ and $\dot{y} = J(q)\dot{q}$. Here $J(:= \frac{dH}{dq})$ is called the manipulator Jacobian matrix.

For the purpose of this paper we will assume that the manipulator is constrained to move in the region

$$Py \geq d. \quad (2)$$

Here $P \in \mathbb{R}^{1\times m}$ and $d \in \mathbb{R}$. We will assume throughout that $P \neq 0$. The linear, single constraint case will allow us to focus on the presentation of our framework. (For a treatment of the multiple nonlinear constraint case we refer to [8].) The boundary set $\{y \in \mathbb{R}^m \mid Py = d\}$ models a hard environmental constraint (or a mathematical constraint).

There are at least two ways to combine the manipulator model (1) with the constraint region (2). The first is to use the kinematic equations to transform (2) to an inequality on joint level. This leads to a fully nonlinear model of a constrained mechanical system. This case is discussed in detail in [8]. A second way, which we will follow here, is to transform manipulator model (1) to a model in terms of Cartesian coordinates. This is in itself an appealing way to proceed since contact is assumed to occur at end-effector level.

The following assumptions are made.

Assumption 2.1

(a) The inertial matrix $M(q)$ is nonsingular.

(b) The Jacobian matrix $J(q)$ is nonsingular in singularity free regions of the workspace.

(c) Contact with the boundary set is made in a singularity free region.

A mathematical model on end-effector level can now be derived from (1).

$$M(y)\ddot{y} + N(y,\dot{y}) = u. \quad (3)$$

Here $M := J^{-T}M_bJ^{-1}, N_b := J^{-T}N_b - J^{-T}M_bJ^{-1}\dot{J}J^{-1}\dot{y}$, and $u := J^{-T}\tau$, where we have suppressed the arguments. We will assume that the control u is locally integrable.

It will prove to be advantageous to study the contact problem in a first-order setting. Let $\mathcal{X}$ denote the state-space. A first-order model of system (3), with $x = [x_1^T, x_2^T]^T := [y^T, \dot{y}^T]^T$, reads:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -M^{-1}(x_1)N(x) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(x_1) \end{bmatrix} u. \quad (4)$$

Define $C := [P \ \ 0]$. In a first-order setting the inequality constraint (2) reads:

$$Cx \geq d. \quad (5)$$

The combination of equation (4) and inequality (5) will be referred to as a constrained mechanical system Σ^c:

$$\Sigma^c : \begin{cases} \dot{x} &= f(x) + g(x)u \\ Cx &\geq d, \end{cases} \quad (6)$$

with the obvious definition of the maps f and g from (4). Σ^c is a complex hybrid dynamical system with continuous-time and discrete event characteristics. It is remarked that (6) is not a state representation of the constrained mechanical system. One still needs to define what constitutes a solution of Σ^c.

Usually, at this point in the modelling procedure the physics of the contact is incorporated into the modelling equations, leading for example directly to the Lagrange multiplier approach to constrained mechanical systems. We will follow a different approach. First, we will investigate what can be deduced already from equations (6). It is only after we have done this that we will incorporate the physics of the problem at hand. The advantages of this approach will become clear in the sequel.

We will need some concepts from the differential geometric approach to nonlinear systems. Let the notation $L_f h(x)$ indicate the directional derivative of the map $h(x)$ along the vector field $f(x)$. The relative degree of a system, denoted by r_0, equals the inherent number of integrations between inputs and an output $y := h(x)$. It is easy to see that for system (6) there holds

$$L_g(Cx - d) = 0 \text{ and } L_gL_f(Cx - d) \neq 0. \quad (7)$$

These equations indicate that for the mechanical system (4), $r_0 = 2$, with respect to an imaginary output $y = Cx - d$. Important will also be the maximal controlled invariant distribution contained in the boundary set of restriction (5). We obtain from [4]:

Lemma 2.2 *The integral manifold of the maximal controlled invariant distribution contained in* $\ker(d(Cx-d))$ *is given by* $\mathcal{N}^* = \{x \in \mathbb{R}^n \mid Cx - d = 0$ *and* $L_f(Cx-d) = 0\}$.

3 Contact and release sets

We study the contact problem by first investigating the interaction between trajectories of the end-effector of the manipulator, modelled by (4), with the boundary set of restriction (5). The following assumption is made.

Assumption 3.1 *In the contact point between end-effector and target there are forces but no moments.*

This assumption implies that impact occurs at a single point, which is unable to transmit a local moment. This is further justified in [6].

First we introduce the following sets to indicate whether of not inequality (5) is satisfied:

$$\mathcal{X}_g := \{x \in \mathbb{R}^n \mid Cx > d\} \tag{8}$$

$$\mathcal{C}_b := \{x \in \mathbb{R}^n \mid Cx = d\} \tag{9}$$

$$\mathcal{X}_f := \{x \in \mathbb{R}^n \mid Cx < d\} \tag{10}$$

Let $\underline{x}$ denote a trajectory of system (4). Furthermore, let $\underline{u} : \mathbb{R} \mapsto \mathcal{U}^{\mathbb{N}}$; i.e. $\underline{u}$ is a countable dimensional vector whose elements take their values in $\mathcal{U}$. Let $\mathcal{X}_{con}$ denote all points where contact can be made with the boundary of the constraint set. Likewise, let $\mathcal{X}_{rel}$ denote all points where trajectories can leave the boundary of the constraint set. These sets are defined formally by:

Definition 3.2 ([8]) *The contact and release sets.*
(i) $\mathcal{X}_{con} := \{x \in \mathcal{C}_b \mid \exists \underline{x}$ *and* $\exists t^* < 0,$ *such that* $\underline{x}(0) = x,$ *and* $\underline{x}(\tau) \in \mathcal{X}_g, \forall \tau : t^* < \tau < 0\}$.
(ii) $\mathcal{X}_{rel} := \{x \in \mathcal{C}_b \mid \exists \underline{x}$ *and* $\exists t^* > 0$ *such that* $\underline{x}(0) = x,$ *and* $\underline{x}(\tau) \in \mathcal{X}_g, \forall \tau : 0 < \tau < t^*\}$.

Note that the contact set $\mathcal{X}_{con}$ says something on the past of a trajectory, whereas $\mathcal{X}_{rel}$ says something about the future. It is shown in [8] that $\mathcal{X}_{con}$ and $\mathcal{X}_{rel}$ switch roles for the time-reversed system. It will prove useful to introduce some additional notation. Define $h_i : \mathbb{R}^n \times \mathcal{U}^{\mathbb{N}} \mapsto \mathbb{R}$ and $r_b(x) : \mathcal{C}_b \mapsto \mathbb{N}$ as

$$h_i(x, \underline{u}) := L_f^i(Cx - d) + \sum_{j=1}^{i} L_g L_f^{j-1}(Cx - d)\underline{u}_{i-j} \tag{11}$$

$$r_b(x) := \min\{i \in \mathbb{N} \mid h_i(x, \underline{u}) \neq 0, \underline{u} \in \mathcal{U}^{\mathbb{N}}\} \tag{12}$$

From (7) it follows that $r_b(x) \leq 2$ for system (4).

We will need the following sets in the sequel.

Definition 3.3 *The sets* $\mathcal{X}_{con,v}$ *and* $\mathcal{X}_{rel,v}$ *(v for velocity), and the set* $\mathcal{N}_c$ *(c for control).*
(i) $\mathcal{X}_{con,v} := \{x \in \mathcal{X}_{con} \mid r_b(x) = 1\}$
(ii) $\mathcal{X}_{rel,v} := \{x \in \mathcal{X}_{rel} \mid r_b(x) = 1\}$
(iii) $\mathcal{N}_c := \{x \in \mathcal{C}_b \mid r_b(x) = 2\}$

The following result is a special case of a result in [8].

Theorem 3.4 *The following holds:*
(a) $\mathcal{X}_{con} = \mathcal{X}_{con,v} \cup \mathcal{N}_c$.
(b) $\mathcal{X}_{rel} = \mathcal{X}_{rel,v} \cup \mathcal{N}_c$.
(c) $\mathcal{C}_b = \mathcal{X}_{con} \cup \mathcal{X}_{rel}$.
(d) $\mathcal{X}_{con,v} \cap \mathcal{X}_{rel,v} = \emptyset$.

Proof: Follows from [8] and definition 3.3. □

We will need the following result.

Lemma 3.5 *The subsets* $\mathcal{X}_{con,v}$*,* $\mathcal{X}_{rel,v}$*, and* $\mathcal{N}_c$ *are invariant under the regular static state feedback* $u = \alpha(x) + \beta(x)v$*, with* v *the new control, for the system (4).*

Proof: Follows from the definitions in (11) and (12), and definition 3.3. □

Let us now apply a preliminary feedback loop that linearizes the equation of motion (4) following the procedure in [10]. (It is assumed that we can measure the positions and velocities.) Take arbitrary, constant matrices A_{21} and A_{22} of appropriate dimensions. Define [10]:

$$u := M(x_1)(M^{-1}(x_1)N(x) + A_{21}x_1 + A_{22}x_2 + w), \tag{13}$$

where w denotes the new control. Then the linear system equations become:

$$\dot{x} = Ax + Bw := \begin{bmatrix} 0 & I \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} w. \tag{14}$$

Note that the constraint equation itself is not affected. As we are now in a linear setting we can apply the results presented in [7].

Proposition 3.6 ([7]) *The following holds:*
(i) $\mathcal{X}_{con,v} = \{x \in \mathcal{C}_b \mid CAx < 0\}$,
(ii) $\mathcal{X}_{rel,v} = \{x \in \mathcal{C}_b \mid CAx > 0\}$,
(iii) $\mathcal{N}_c = \{x \in \mathcal{C}_b \mid CAx = 0\}$.

It is not difficult to prove that $\mathcal{N}^* = \mathcal{N}_c = \mathcal{V}^*$, where $\mathcal{V}^*$ is the largest controlled invariant subset of system (14) that is contained in the boundary set of (5). If we now view the boundary set as a mathematical constraint rather than a hard environmental constraint, it follows that in $\mathcal{X}_{con} \backslash \mathcal{V}_c$ application of a smooth control can not prevent a trajectory of system (4) to enter $\mathcal{X}_f$. It is clear that this finding has consequences for (feedback) controller design if one of the objectives is smooth contact with the boundary set. The above leads to the concept of forbidden controls, which will prove useful when the collision maps are discussed. A formal definition is given next.

Definition 3.7 *Let* $x \in \mathcal{N}_c$. *The set of (locally) forbidden controls is defined as:* $\mathcal{U}_f(x) = \{u \in \mathcal{U} \mid \exists \underline{x}$ *of system (4) and* $\exists t^* > 0$ *such that* $\underline{x}(0) = x$ *and* $\underline{x}(\tau) \in \mathcal{X}_f, \forall \tau : 0 < \tau < t^*\}$.

4 The collision map

In this section we will introduce collision maps in our framework that will make (5) an invariant set for (4).

Assumption 4.1 *Collisions are instantaneous.*

Suppose that at some point in time a state trajectory intersects the boundary set, say at time t_*. If, for instance, $\underline{x}(t_*) \in \mathcal{X}_{con,v}$ then, whatever (smooth) control is applied, the motion can not remain in $\mathcal{X}_g \cup \mathcal{C}_b$. In $\mathcal{N}_c$ it depends also on the control whether a trajectory will proceed in $\mathcal{X}_g$ or will remain in $\mathcal{C}_b$. This motivates the following definitions.

Definition 4.2 *Let $x \in \mathcal{C}_b$. Then we will say that a collision at x is an uncontrolled, controlled, collision if $x \in \mathcal{X}_{con,v}$, $x \in \mathcal{N}_c$, respectively. Likewise, we will say that a release is an uncontrolled, controlled, release if $x \in \mathcal{X}_{rel,v}$, $x \in \mathcal{N}_c$, respectively.*

From proposition 3.6 and definition 4.2 the following result is now immediate. (The proof is omitted.)

Lemma 4.3 *For the constrained mechanical system (6) uncontrolled collisions occur in the set $\mathcal{X}_{con,v} = \{(y, \dot{y}) \,|\, Py = d, P\dot{y} < 0\}$, uncontrolled release takes place in the set $\mathcal{X}_{rel,v} = \{(y, \dot{y}) \,|\, Py = d, P\dot{y} > 0\}$, and controlled contact and release can take place in the set $\mathcal{N}_c = \{(y, \dot{y}) \,|\, Py = d, P\dot{y} = 0\}$.*

It is important to remark that in our framework the fact whether or not a collision is a controlled collision follows already from the models for the unconstrained manipulator model, and the model of the boundary set. The physics of contact itself need not to be incorporated: (un)controlled collisions are a feature of the model. In fact, the following important result holds:

Proposition 4.4 *The contact and release sets for a constrained mechanical system (6) are independent of the parameters of the unconstrained mechanical system model.*

Proof: Follows from lemma 4.3. □

From the proposition it follows that for the contact and release sets it is the second-order nature of a mechanical system that is important, and not the parameters of the model. As a consequence we have that uncontrolled contact, i.e. collisions, do occur. In our point of view, a discussion on collisions should thus be an integral part of a treatment of constrained mechanical systems.

For a manipulator, plastic collisions indicate that the end-effector remains on the boundary set. An elastic collision indicates that some energy can be taken from the system, but a bounce still happens. This, and lemma 4.3, motivates the following definitions of the collision map(s).

Definition 4.5 *The collision map $T : \mathcal{X}_{con} \mapsto \mathcal{X}_{rel}$.*
(a) The collision map for uncontrolled elastic collisions is defined as a map: $T_v : \mathcal{X}_{con,v} \mapsto \mathcal{X}_{rel,v}$.
(b) The collision map for uncontrolled plastic collisions is defined as a map: $T_p : \mathcal{X}_{con,v} \mapsto \mathcal{N}_c$.
(c) The collision map for controlled collisions is defined as a map: $T_c : \mathcal{N}_c \times \mathcal{U} \mapsto \mathcal{N}_c \times (\mathcal{U} \backslash \mathcal{U}_f)$.

We have now reserved a place to model collisions with the boundary set. The exact definition of the map T must be determined by external physical modelling. We merely advocate that collisions can be discussed already on a general system theoretical level. The collision map will be used in the remainder to make the constraint set an invariant set for the mechanical system.

It can be seen that plastic collisions differ in a number of ways from elastic collisions, especially when control enters the formulation. Notably, if $x \in \mathcal{X}_{con} \backslash \mathcal{N}_c$ is mapped into $\mathcal{N}_c$, then control enters the formulation. This can be captured by applying the map T_c immediately after the map T_p. Of course, one could also combine the latter two maps in one controlled collision map, but the present definitions allow us to unify elastic and inelastic collisions when discussing the behaviour of constrained dynamical systems below. Use of the collision maps allows us to state the following result.

Theorem 4.6 *The constrained system Σ^c given in (6) can alternatively be presented as:*
$\Sigma^c = \{\underline{x} : \mathbb{R} \mapsto \mathcal{X} | \exists u$ piece-wise C^∞ such that $\forall t_0 \in \mathbb{R}$:
(i) $\underline{x}(t_0) \in \mathcal{X}_g \cup \mathcal{C}_b$;
(ii) $\underline{x}(t) \in \mathcal{X}_g \cup \mathcal{X}_{rel}, t \geq t_0 \Rightarrow \dot{\underline{x}}(t) = f(\underline{x}(t)) + g(\underline{x}(t))u(t)$;
(iii) $\underline{x}(t) \in \mathcal{X}_{con} \backslash \mathcal{N}_c$, $t \geq t_0 \Rightarrow \lim_{t^* \downarrow t} \underline{x}(t^*) = T(\underline{x}(t))$;
(iv) $\underline{x}(t) \in \mathcal{N}_c$, $t \geq t_0 \Rightarrow \lim_{t^* \downarrow t} (\underline{x}, \underline{u})(t^*) = T_c(\underline{x}(t), \underline{u}(t))$, with $\underline{u}_i(t) := u^{(i)}(t)$ }.

Proof: Follows from theorem 3.4 and the analysis in this section, with the collision map T as in definition 4.5. □

From theorem 4.6 it follows that trajectories of the constrained mechanical system consist of concatenated path pieces of the unconstrained system, where the collision map is used to deal with the local behavior in case of collisions. For the global behavior one still has to take into account infinite series of impacts in a finite time. For instance, an infinite series of impacts, at times $t_n (n \to \infty)$, in a finite time, say t^*, can be modelled by appropriate choice of the collision maps such that $\lim_{t_n \to t^*} T_v(\underline{x}(t_n)) = T_p(\underline{x}(t^*))$. We refer to [1] for some more details on this type of collisions for mechanical systems.

5 The physics of contact

Until now we have used the models of the unconstrained mechanical system and the boundary set of the constraint to come to a system theoretical framework to discuss the contact problem. In this section we will incorporate the physics of the problem at hand and compare our framework with constrained mechanical system models.

Assuming that contact is frictionless, the following model is taken from [2], and adapted to our notation:

$$M(y)\ddot{y} + N(y, \dot{y}) = v + P^T f, \tag{15}$$

$$f = \begin{cases} 0 & \text{during free motion,} \\ \Gamma & \text{during collision,} \\ \lambda & \text{during constrained motion,} \end{cases} \tag{16}$$

combined with equation (2). Here $P^T f$ represents the contact force matrix, Γ denotes the impulsive force due to collisions, and λ denotes the Lagrange multiplier which has the dimension of the force for a holonomic equality constraint $Py = d$.

We will show that this model fits our framework by special choices of the collision map T. Firstly, in the free motion phase it follows from theorem 4.6 that model (15) and our model coincide. Secondly, for the (uncontrolled) collision phase it follows from definition 4.5 and lemma 4.3 that the collision map T_v yields a jump in (part of) the velocity. For systems with nonzero mass, this implies a discontinuity in momentum, which can only be caused by an impulsive force. The following result can also be found in [2, theorem 1], where the proof uses the concept of virtual masses to arrive at the result (for nonlinear constraints).

Proposition 5.1 *Let $\dot{y}_n$ denote the velocity component normal to the boundary set of constraint (2) at the moment of collision. Then there is no uncontrolled collision ($\Gamma = 0$) if and only if $\dot{y}_n = 0$.*

Proof: From lemma 4.3 follows that there are no collisions if and only if $Py = d$ and $P\dot{y} = 0$. Hence, $\dot{y} \in \ker(P)$, i.e. $\dot{y}_n = 0$. The collision map T can be taken as the identity on $\mathcal{X}$, meaning that there is no discontinuity in the state. Consequently, $\Gamma = 0$. □

In case contact is made in $\mathcal{X}_{con,v}$ an impulse does enter the system. Knowledge of the collision map T is a prerequisite to obtain an expression for the impulse Γ, since the velocity characteristics on end-effector level immediately after the collisions are necessary. The importance of (the estimate of) Γ is a consequence of the fact that a small impulsive force leads (in general) to small changes in the velocities.

The most widely used collision map is known as Newton's rule, which relates the velocity immediately after contact to the velocity before impact using a restitution coefficient ϵ. This rule can be generalised to higher dimensions (see e.g. [1]). In our framework this can be captured by explicitly defining T_v as: $T_v(y, \dot{y}) = (y, \dot{y}_1 - \epsilon\dot{y}_2)$ where

$$\dot{y}_1 := (I - M^{-1}P^T(PM^{-1}P^T)^{-1}P)\dot{y}, \qquad (17)$$
$$\dot{y}_2 := M^{-1}P^T(PM^{-1}P^T)^{-1}P\dot{y}. \qquad (18)$$

Hence, by choice, $\dot{y}_1$ is the skew projection of $\dot{y}$ on $\ker(P)$ and $\dot{y}_2$ is a projection of $\dot{y}$ on $\mathrm{im}(M^{-1}P^T)$ for system (3). It is clear that the impulse now only changes the velocity components in $\mathrm{im}(M^{-1}P^T)$ for system (3), i.e. the impulse can be modelled as $P^T\Gamma$ for system (15).

Finally, for the constrained motion phase it is assumed that $Py = d$ on an interval $[t_1, t_2]$. System (6) reduces to a singular system. Define as controlled collision map: $T_c(x, u) := (x, v + P^T\lambda)$ (c.f. definition 4.5), where λ must be chosen such that whatever the control v, the control $u := v + P^T\lambda$ is not forbidden, i.e. $u \notin \mathcal{U}_f(x)$ (c.f. definition 3.7). Note that the state is left invariant under the map T_c. By continuous application of the map T_c on the interval $[t_1, t_2]$ one arrives at model (15). An explicit expression of λ, reads ([8, 10, 12]): $\lambda = -(PM^{-1}P^T)^{-1}PM^{-1}(v - N)$.

It is concluded that our approach to restricted systems fits the approach generally taken in the literature.

6 On impact control

There are at least two control methodologies that have reached a certain level of maturity for constrained mechanical system, i.e. pure force control and impedance control. Due to space limitations we will only discuss (briefly) impedance control where the boundary set $\{y \in \mathbb{R}^m \mid Py = d\}$ models a very stiff surface. The term impedance control typically refers to a control law that implements some target dynamics consisting of selected inertial, damping, and stiffness parameters. Let the target dynamics be given by [11, 15]:

$$M_d\ddot{y} + D_d(\dot{y} - \dot{y}_d) + K_d(y - y_d) = F. \qquad (19)$$

Here M_d, D_d and K_d are $(m \times m)$ positive-definite matrices representing the desired inertia, damping, and stiffness of the closed loop system, y_d represents the desired end-effector position, and F denotes the Cartesian force exerted on the environment.

Solving (19) for $\ddot{y}$ and substitution into (15) yields the impedance control law. A simplified controller can be obtained by setting $M_d = \rho M$, $\rho \in \mathbb{R}$. This gives [11, 15]:

$$v = -(1 - \frac{1}{\rho})F + \frac{1}{\rho}(D_d(\dot{y} - \dot{y}_d) + K_d(y - y_d)) + R, \quad (20)$$

where R contains terms that do not depend on ρ.

The factor $(1 - \frac{1}{\rho})$ plays the most important role in determining the magnitude of impact, because the impact force is much larger than all other forces on the right-hand side of the above equation. If $\rho = 1$ the first term in (20) vanishes and the control force at impact is reduced since this choice blocks the force signal from feeding back into the controller [11], i.e. force feedback is disabled. In [11] it is remarked that simulations show that for stiff environments, the choice $\rho = 1$ provides a 'good compromise between rebound effects and the magnitude of the impulsive forces', but also with the reduced impedance controller the manipulator rebounced.

At first sight it may seem strange that not using force sensor information in the feedback loop is more stable than the case where one does use this sensor information. However, from our analysis it follows that there is a good mathematical reason for doing so: in $\mathcal{X}_{con,v}$ impact can not be controlled by a smooth control and rebound effects are unavoidable, also for very stiff surfaces. This is supported also by results reported in for instance [16], where during impact simulations with a steel pedestal at least one bounce always occurred. One could apply an impulse to the system to counteract the impulse that arises due to impact, but this is not a practical solution. Carefully manipulating the impedance control laws such that the measured force is not fed back is a sensible thing to do, and is motivated also by our analysis.

We conclude that control design should be aimed at making contact with small velocities, and at effective avoidance of multiple bounces.

7 Concluding remarks

In this paper we have presented a framework to study the contact problem for linear continuous-time mechanical systems subject to unilateral constraints. Our analysis amounts to a top-down approach to investigate the effect (of the boundary set) of a unilateral constraint on the behaviour of a mechanical system. Our main results are a system theoretical framework in which we described exactly what happens upon contact, an identification of the specific places for modelling the laws of collision, and a precise definition of the constrained state-space system in terms of the restricted behaviour.

We have shown that analysis of impact phenomena benefits from a discussion of the mathematical models on a system theoretical level before the physics of contact are incorporated. We have also shown that the velocity of the end-effector of the manipulator at the moment of contact and the (unit) vector normal to the plane of impact play an important role in the impact phase. The parameters of the model of the mechanical system do not have an influence on the sets where contact and release (can) take(s) place. It is the second-order nature of the model that is important. The inertia matrix of the manipulator model does play a role when the magnitude of the resulting impulse must be calculated. Inclusion of the physics of contact amounts to choosing the appropriate collision maps. For instance, friction can be included in our framework by further detailing of equation (17). This is left as part of future research.

It is stated in [9] that for several control methods 'the suggested improvements are inconclusive unless considerations of hardware, software and equipment specific data are taken into account'. The analysis performed in this paper can be seen to extent this statement to the impact phase of mechanical systems subject to unilateral constraints.

References

[1] B. Brogliato, S-I. Niculescu and P. Orphant, On the control of finite-dimensional mechanical systems with unilateral constraints, *Preprint*, 1996.

[2] L. Cai and A.A. Goldenberg, Robust control of position and force for robot manipulators involving both free and constrained motion, In: *Proceedings of the 29th Conference on Decision and Control*, Honolulu, Hawaii, pp. 1943 - 1948, 1990.

[3] B. Chapnik, G.R. Heppler and J.D. Aplevich, Modelling impact on a one-link flexible robotic arm, *IEEE Transactions on Robotics and Automation*, Vol. 7, No. 4, pp. 479-488, 1991.

[4] N.H. McClamroch, Feedback stabilization of control systems described by a class of nonlinear differential-algebraic equations. *Systems & Control Letters*, 15, pp. 53-60, 1990.

[5] G. Colombia, F. Didot, G. Magnani and A. Rusconi, External servicing testbed for automation and robotics, *IEEE Robotics & Automation Magazine*, Vol. 3, No. 1, pp. 13-23, 1996.

[6] X. Cyril, G. Jaar, M. Ingham, and A.K. Misra, Post-impact dynamics of a spacecraft-mounted manipulator, 44th Congress of the International Astronautical Federation, October 16-22, Graz, Austria, 1993.

[7] A.A. ten Dam, E. Dwarshuis and J.C. Willems, The contact problem for linear continuous-time dynamical systems: a geometric approach, To appear.

[8] A.A. ten Dam and J.C. Willems, Constrained nonlinear dynamical systems: a differential geometric approach to the contact problem, In preparation.

[9] A.A. Goldenberg, Analysis of force control based on linear models, In: *Proceedings of the 1992 IEEE International conference on Robotics and Automation*, Nice, France, May 1992, pp. 1348-1353.

[10] A. J. Koivo, *Fundamentals for Control of Robotic Manipulators*, John Wiley & Sons, 1989.

[11] Z.C. Lin, R.V. Patel, and C.A. Balafoutis, Impact reduction for manipulators using augmented impedance control, *Journal of Robotic Systems*, 12 (5), pp. 301-313, 1995.

[12] P. Lötsted, Mechanical systems of rigid bodies subject to unilateral constraints, *Siam J. Appl. Math.* Vol. 41, no. 2, pp. 281-296, 1982.

[13] N. Mandal, and S. Payandeh, Control for robotic contact tasks: an experimental study, *Journal of Robotic Systems*, 12 (1), pp. 67-92, 1995.

[14] J.K. Mills and D.M. Lokhorst, Control of robotic manipulators during general task execution: a discontinuous control approach, *The International Journal of Robotics Research*, Vol. 12, No. 2, pp. 146-163, 1993.

[15] W.S. Newman and M.E. Dohring, Augmented impedance control: an approach to compliant control of kinematically redundant manipulators. In: *Proceedings of the IEEE Conference on Robotics and Automation*, pp. 30-35, 1991.

[16] R. Volpe and P. Khosla, A theoretical and experimental investigation of impact control for manipulators, *The International Journal of Robotics Research*, Vol. 12, No. 4, pp. 351-365, 1993.