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Adaptive Generation of Structured Grids
Part I: Introduction and State-of-the-art

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ABSTRACT
The concept of structured grid adaptation is introduced and the one-dimensional equidistribution principle for grid adaptation is presented. Several formulations of this principle are linked together which is used to present a brief overview of the "State-of-the-art" from the literature. This part of the lecture is partially based on reference [1].

Adaptive Generation of Structured Grids, Part1

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Abstract

The concept of structured grid adaptation is introduced and the one-dimensional equidistribution principle for grid adaptation is presented. Several formulations of this principle are linked together which is used to present a brief overview of the "State-of-the-art" from the literature. This part of the lecture is partially based on reference [1].

Contents

1 Introduction

Computational grids are used for the discretisation of partial differential equations (PDE's) such that solution of the discrete system of algebraic equations provides an approximation of the solution of the system of PDE's. The accuracy of the solution approximation depends on:

1. the computation method used,
2. the computational grid, and
3. the solution itself.

The computation method at one hand determines which quantities determine the truncation error and the computational grid and the solution itself at the other hand determine the magnitude of these quantities.

Let for example the function $x(\xi)$ determine a non-uniform grid in one-dimensional space as the image of a uniform grid in one-dimensional space:

$$x_i = x(\xi_i), \quad i = 1, ..., N; \quad \xi_i = i/(N-1). \tag{1}$$

An approximation of the first derivative of a function $f(x)$ with respect to x is:

$$f_x = f_\xi \xi_x = f_\xi / x_\xi = \frac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}} + T.E. \tag{2}$$

The truncation error $T.E.$ is $O((N-1)^{-2})$ and the leading term in $T.E.$ is a function of both $x(\xi)$ and $f(x)$.

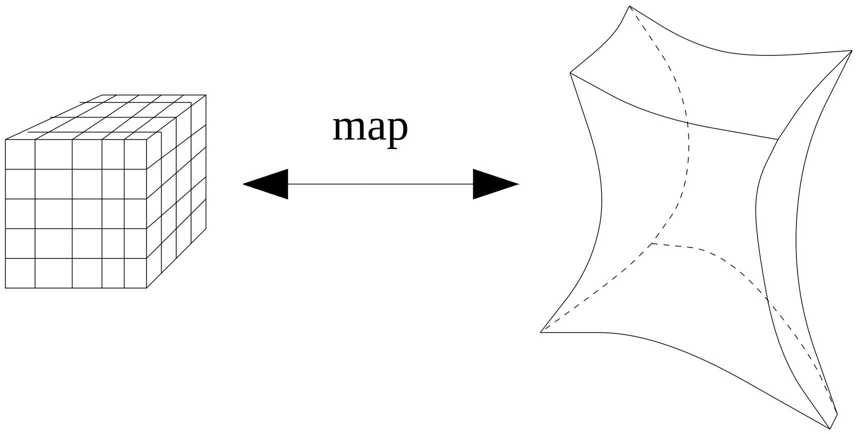

Figure 1: Definition of single-block structured grid.

The fact that the solution of the problem at hand is incorporated in the truncation error prohibits *a priori* tuning of the grid point locations other than global refinement (addition of grid points) or tuning to an estimate of the solution. As an alternative a typical computation could start on a given initial grid and during the course of computation the grid could be adjusted. This explicitly demonstrates the subject of the present paper: adaptation of the current computational grid with respect to the current approximate solution of the problem at hand with the goal to increase the computation accuracy.

The present paper deals with adaptation of so-called single-block structured grids. A single-block structured grid is defined as the image of a uniform grid in $[0,1]^n$ ($n = 1$ denotes the unit interval, $n = 2$ denotes the unit square and $n = 3$ denotes the unit cube) under a given regular map to the physical domain, see Fig. 1. By a regular map we mean that it is continuous, one-to-one and invertible. Due to the definition of a single-block structured grid adaptation can not be carried out by local addition or subtraction of grid points since this would remove the structure of the grid. Global addition of subtraction of grid points is indeed possible but not very efficient. If the current solution of the problem at hand requires local refinement of the grid then global refinement is overdone. The alternative is grid point movement. If local refinement is required grid points can locally be moved towards the particular spot while some distance away grid points could stay at almost fixed positions. Grid adaptation by point (node) movement is by far not as efficient and effective as local grid refinement of so-called unstructured grids, but todays CFD codes for viscous high Reynolds-number flows, however, are based on discretisation of the Navier-Stokes equations on structured grids incorporating highly stretched grid cells with typically aspect ratios of 1:1000 or 1:10000. Therefore it is justified from an industrial point of view to put effort in the development of grid adaptation methods based on node movement within structured grids.
A basic underlying concept for the generation and adaptation of structured grids is to identify grid lines (in 2D) and grid surfaces (in 3D) as iso-lines and iso-surfaces of scalar functions that are defined in the physical domain. Figure 2 shows an example of two families of iso-lines in the unit square, each based on

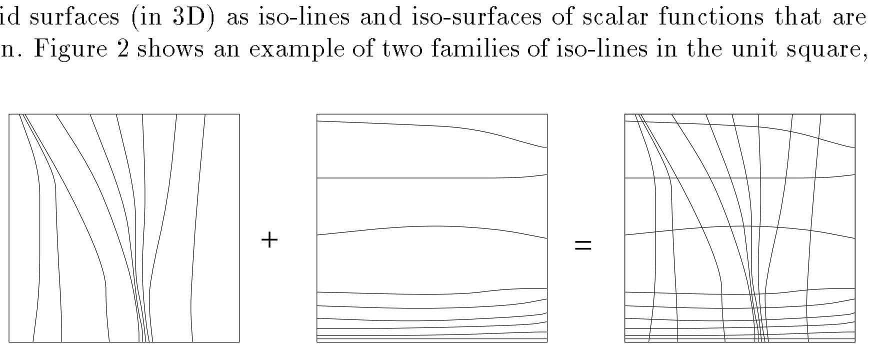

Figure 2: Two families of iso-lines of two different scalar functions combined into a single computational grid.

a scalar function. The structured grid is subsequently constructed by combination of these two families of iso-lines and taking the intersections as grid points. From this concept it is straightforward to define:

Adaptation of structured grids is equivalent to adaptive construction of scalar functions over the domain at hand.

2 One-dimensional Equidistribution Principle

In one-dimensional problems adaptive grids consist of a number of points along the real axis with their locations tuned to the solution of the problem. The goal of the adapted grid is to provide a set of points that are suitable for discretisation of the equations that describe the problem. As is mentioned in the introduction the use of finite differences introduces local truncation errors which depend on both the grid and the solution of the problem at hand. To minimize these truncation errors one can try to tune the mesh size to the local behaviour of the problem solution. Let the variation of the problem solution be weighted in some sense by a weight function, then the following *Equidistribution Principle* can be applied:

$$(weightfunction) * (meshsize) = constant$$

As a result the mesh size will be small where the weight function is large and, vice versa, the mesh size will be large where the weight function is small. This principle plays a basic role in the development of many adaptation methods, also for multi-dimensional problems.

Mathematically the equidistribution principle can be formulated as:

$$w(x)x_\xi = constant \tag{3}$$

where $x(\xi) : [0,1] \mapsto [0,L] \in R$ is the map that controls the adapted grid and $w(x) : [0,L] \in R \mapsto D \in R^+$ is the weight function. Once the adaptive map $x(\xi)$ is obtained by solving equation (3) the adaptive grid is constructed as the image of a uniform grid on the unit interval under the map $x(\xi)$, see equation (1). Examples of weight functions incorporating first and/or second derivatives of the problem solution are presented and discussed by Thompson [2]. One of these examples incorporates the following weight function:

$$w(x) = \sqrt{1 + f_x^2}, \tag{4}$$

where $f(x)$ is the problem solution. It is easily verified that substitution of $w(x)$ given by expression (4) into the equidistribution principle (3) yields:

$$\sqrt{x_\xi^2 + f_\xi^2} = constant, \tag{5}$$

meaning that grid points are equally distributed over the graph $(x, f(x))^T$ in the two-dimensional plane. In case $f(x)$ is a monotonically increasing function of x, another interesting example is to take:

$$w(x) = f_x. \tag{6}$$

It is again easily verified that substitution of $w(x)$ given by expression (6) into the equidistribution principle (3) yields:

$$f_\xi = constant, \tag{7}$$

meaning that f has become a linear function of ξ.

Eiseman [3] presented a dozen ways to formulate the equidistribution principle, ranging from the differential statement given by (3) to variational statements. We will recall a selection of these various formulations that is representative for many adaptation methods presented in the literature.

- Evolutionary Statement

$$\frac{\xi_i^{k+1}}{\langle w \rangle^i} = \frac{1}{2}\left(\frac{\xi_{i-1}^k}{w^{i-\frac{1}{2}}} + \frac{\xi_{i+1}^k}{w^{i+\frac{1}{2}}}\right); \quad \frac{1}{\langle w \rangle^i} = \frac{1}{2}\left(\frac{1}{w^{i-\frac{1}{2}}} + \frac{1}{w^{i+\frac{1}{2}}}\right), \tag{8}$$

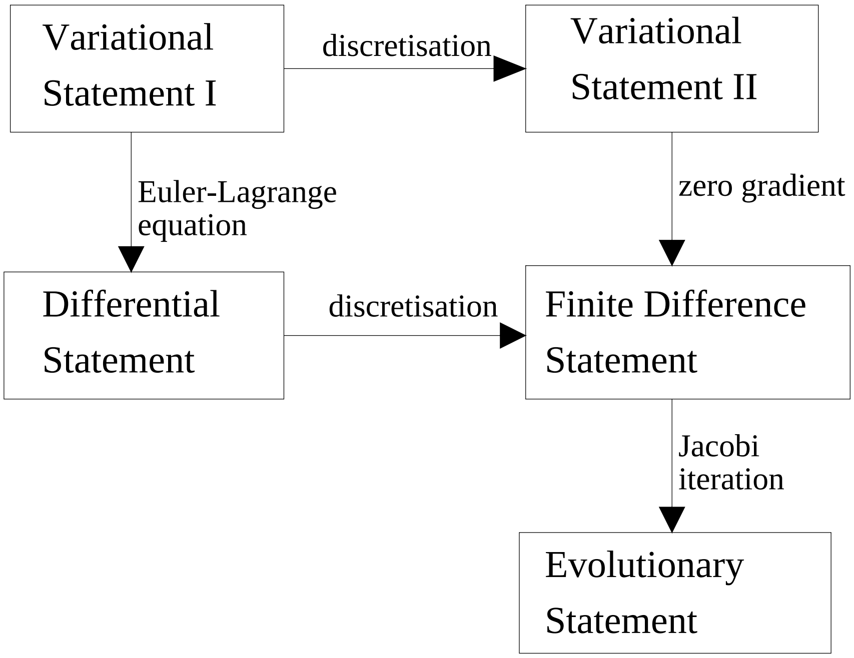

Figure 3: Interrelationship between various formulations of the *Equidistribution principle*

- Finite Difference Statement

$$\frac{\xi_{i+1}-\xi_i}{w^{i+\frac{1}{2}}} - \frac{\xi_i-\xi_{i-1}}{w^{i-\frac{1}{2}}} = 0, \tag{9}$$

- Differential Statement

$$\frac{d}{dx}\frac{\xi_x}{w} = 0, \tag{10}$$

- Variational Statement I

$$minimize \quad K[\xi] = \frac{1}{2}\int_\Omega \frac{\xi_x^2}{w}dx, \tag{11}$$

- Variational Statement II

$$minimize \quad F(\xi_1,...,\xi_N) = \frac{1}{2}\sum_i \frac{(\xi_{i+1}-\xi_i)^2}{w^{i+\frac{1}{2}}}. \tag{12}$$

All of these statements are interrelated to eachother as is illustrated in Fig. 3. Starting from Variational Statement I we have two possibilities: either we discretise the integral and obtain Variational Statement II, or we derive the associated Euler-Lagrange equation and arrive at the Differential Statement. Then the Finite Difference Statement is obtained by either discretising the Differential Statement or derive the zero-gradient condition to satisfy the Variational Statement II. Finally the Evolutionary Statement is obtained by applying Jacobi iteration to the Finite Difference Statement.

3 Two-dimensional extension

In view of the observation that grid generation or adaptation in two (or three) dimensions can be considered as the construction of two (or three) suitable scalar functions, see Fig. 2, the objective of the present section is to:

Find computational coordinates $\xi(x,y)$ and $\eta(x,y)$ such that iso-ξ and iso-η lines projected on the physical domain at hand form a smooth grid that is adapted to some given scalar function, say $\rho(x)$.

Without loss of generality we assume that the physical domain at hand, say Ω, is the unit square and that the initial grid is a uniform grid with square cells. Like with the formulation of the one-dimensional *Equidistribution principle* we can define a number of statements for the generation of an adapted grid in Ω:

- Evolutionary Statement

$$\frac{\boldsymbol{\xi}_{ij}^{k+1}}{\langle w\rangle^{ij}} = \frac{1}{4}\Big(\frac{\boldsymbol{\xi}_{i-1,j}^{k}}{w_1^{i-\frac{1}{2},j}} + \frac{\boldsymbol{\xi}_{i,j-1}^{k}}{w_2^{i,j-\frac{1}{2}}} + \frac{\boldsymbol{\xi}_{i+1,j}^{k}}{w_1^{i+\frac{1}{2},j}} + \frac{\boldsymbol{\xi}_{i,j+1}^{k}}{w_2^{i,j+\frac{1}{2}}}\Big), \quad \boldsymbol{\xi} \equiv (\xi,\eta)^T, \tag{13}$$

with

$$\frac{1}{\langle w\rangle^{ij}} = \frac{1}{4}\Big(\frac{1}{w_1^{i-\frac{1}{2},j}} + \frac{1}{w_2^{i,j-\frac{1}{2}}} + \frac{1}{w_1^{i+\frac{1}{2},j}} + \frac{1}{w_2^{i,j+\frac{1}{2}}}\Big), \tag{14}$$

- Finite Difference Statement

$$\Big\{\frac{\boldsymbol{\xi}_{i+1,j} - \xi_{i,j}}{w_1^{i+\frac{1}{2},j}} - \frac{\boldsymbol{\xi}_{i,j} - \xi_{i-1,j}}{w_2^{i-\frac{1}{2},j}}\Big\} + \Big\{\frac{\boldsymbol{\xi}_{i,j+1} - \xi_{i,j}}{w_1^{i,j+\frac{1}{2}}} - \frac{\boldsymbol{\xi}_{i,j} - \xi_{i,j-1}}{w_2^{i,j-\frac{1}{2}}}\Big\} = 0, \quad \boldsymbol{\xi} \equiv (\xi,\eta)^T, \tag{15}$$

- Partial Differential Statement

$$\frac{\partial}{\partial x}\Big(\frac{\xi_x}{w_1}\Big) + \frac{\partial}{\partial y}\Big(\frac{\xi_y}{w_2}\Big) = 0, \quad \frac{\partial}{\partial x}\Big(\frac{\eta_x}{w_1}\Big) + \frac{\partial}{\partial y}\Big(\frac{\eta_y}{w_2}\Big) = 0, \tag{16}$$

- Variational Statement I

$$K[\xi,\eta] = \frac{1}{2}\int_\Omega \Big\{\frac{\xi_x^2}{w_1} + \frac{\xi_y^2}{w_2} + \frac{\eta_x^2}{w_1} + \frac{\eta_y^2}{w_2}\Big\} d\Omega, \tag{17}$$

- Variational Statement II

$$F(\xi_{i,j}; i = 1,..,N; j = 1,...,N) = \frac{1}{2}\sum_i \sum_j \Big\{\frac{\|\boldsymbol{\xi}_{i+1,j} - \boldsymbol{\xi}_{i,j}\|^2}{w_1^{i+\frac{1}{2},j}} + \frac{\|\boldsymbol{\xi}_{i,j+1} - \boldsymbol{\xi}_{i,j}\|^2}{w_2^{i,j+\frac{1}{2}}}\Big\} \tag{18}$$

Note that if the two weight functions are both identical to one, all of the above formulations represent the Laplace equations.

It was noted by Thompson [2] that for solution of the computational coordinates $(\xi,\eta)^T$ as functions of the physical coordinates $(x,y)^T$ on general 2D domains we need an appropriate grid. To construct such a grid we need in turn to solve PDE's: hence we have arrived at an "chicken or egg" problem (who was first?). To circumvent this problem Thompson proposed to interchange the roles of the dependent and independent variables. As a consequence the grid equations have to be solved on the rectangular computational domain for the physical coordinates as functions of the computational coordinates. The grid in the computational domain is simply uniform. Thompson evaluated this idea for the Poisson equations:

$$\nabla^2\xi = P, \quad \nabla^2\eta = Q, \tag{19}$$

which can be transformed into:

$$g^{ij}\frac{\partial^2 \mathbf{x}}{\partial\xi^i\partial\xi^j} + P_k\frac{\partial \mathbf{x}}{\partial\xi^k} = 0, \tag{20}$$

where $\mathbf{x} = (x,y)^T$, $\xi^1 = \xi$, $\xi^2 = \eta$, $P_1 = P$, and $P_2 = Q$, and g^{ij} is the contravariant metric tensor:

$$g^{ij} = \frac{\partial \xi^i}{\partial x^k}\frac{\partial \xi^j}{\partial x^k} \tag{21}$$

Note that due to the transformation we have exchanged uncoupled linear PDE's (19) for coupled quasi-linear PDE's (20).

While for the generation of initial grids the above described transformation is necessary to tackle the problem, the situation for grid adaptation is quite different since an initial grid is present which can be used to solve the grid equations. Furthermore weight functions that are involved in the grid adaptation process will be functions of the physical coordinates $(x,y)^T$, e.g., incorporating the density gradient. Hence when the grid adaptation equations are transformed and solved for the physical coordinates as functions of the computational coordinates then evaluation of the weight functions at the updates of the points in physical space requires interpolation of the flow solution at every iteration. Therefore it can be beneficial to solve the linear equations directly for the computational coordinates $(\xi,\eta)^T$ as functions of the physical coordinates $(x,y)^T$ and to re-invert the map numerically to obtain the physical coordinates as functions of the computational coordinates afterwards. This is illustrated in part 2 of this lecture.

4 State-of-the-art from the literature

In this section we present a number of grid adaptation examples from the literature primarily presented during the last five years. It is noted that this is by no means an extensive overview of the literature and the reader is encouraged to consult the excellent overviews of Thompson [4], Eiseman [3],[5], [6] Thompson and Weatherill [7] and the very recent one by Baker [8]. For general theory on grid generation the reader is refered to the standard books of Thompson, Warsi and Mastin [2], and Knupp and Steinberg [9], and the paper on variational formulations by Warsi and Thompson [10].

Concepts The evolutionary approach has been exploited amongst others by Eiseman et. al., e.g. [11], and Pao and Abdol-Hamid [12]. They use the alternate direction adaptation approach which is convenient for dynamic adaptation. Eiseman [13], and Connett et. al., e.g. [14], use the concept of prescribing a displacement for each node based on its position with respect to the surrounding nodes. Benson and McRae [15] use a parametric domain to adapt the grid points based on movement towards a center of "mass".

The finite difference approach has been exploited amongst others by Nakahashi and Deiwert [16]. They use a tension and torsion spring analogy and solve the system of equations to minimize the potential energy in the spring system.

Within the PDE formulations the adaptation approach of Kim and Thompson [17] is a practicle extension of the widely used elliptic grid generation approach of Thompson [18]. Eiseman [3] shows that the PDE formulation is a subset of the PDE formulation that is obtained from an adaptive extension of the variational formulation of the Laplace equations. Adaptation is obtained by incorporation of weight functions in the control functions (source terms) of the Poisson system. The weight functions measure the gradients of the flow solution. In references [19] and [20] the implementation of these concepts into industrial grid generation systems is demonstrated. Lee and Loellbach, e.g. [21], use a parametric domain and adapt the points in response to source potentials. The parametric map is based on the initial grid. After adaptation the points are mapped back to the physical space. Another interesting PDE approach is described by Catherall [22] who developed a mixture of PDE's called the LPE system, using a weighted sum of Laplace, Poisson and Equidistribution equations. In this way user-defined control is obtained over smoothness, resemblence of the initial grid and over adaptation.

Finally within the class of variational methods we start with the functional presented by Winslow [23] which consists of a weighted integration of the gradients of the computational coordinates. The adaptation algorithm is isotropic because a single weight function is used. The functional proposed by Eiseman [3] is an extension of the Winslow functional incorporating two or three weight functions enabling anisotropic adaptation. Brackbill and Saltzman [24] developed a composite functional as the weighted sum of three separate functionals accounting for smoothness, orthogonality and adaptivity respectively resulting in fairly complex PDE's but enabling explicit control over these three grid properties that influence the discretisation errors in the flow solution. Also very interesting is the functional developed by Jaquotte, e.g., [25], which resembles the deformation of elastic material. While direct minimisation of this functional may require considerable computational effort, the recently exploited approach of Coussement [26] based on solution the associated Euler-Lagrange equations is promising. Dvinsky [27] introduced the minimisation of an 'energy' integral resembling the smooth distribution of a grid over the graph of a monitor surface. This can be interpreted as solution of the Laplace equations on the monitor surface described by the Laplace-Beltrami equations. Extension of this method to multiple monitor surfaces is explored by Spekreijse et. al. [28]. Finally a functional enabling anisotropic grid adaptation has been proposed by the present author et. al. [29], [30],[31] which is extensively discussed in the second part of this lecture.

Applications for 3D Euler Some adaptation results for 3D structured grids suitable for solution of the Euler equations have been presented in [17], [32], [33] and [34]. Kim and Thompson [17] compared their control function (Poisson) approach to the variational approach of Brackbill and Saltzman [24] by application on a 3D grid around the ONERA M6 wing. Their conclusion is that both approaches have basically the same potentials but the control function approach should be the more promising tool due to shorter computing times and higher sensitivity to the flow solution. Tu and Thompson [32] apply the control function approach within the EAGLE 3D code to an eight-block finned body of revolution at transonic speeds, obtaining an improved quality of aerodynamics simulation. Catherall [33] and Le Pape et. al. [34] obtain adaptive solutions for the ONERA M6 wing with the LPE method [22] and the Jaquotte functional [25] respectively showing adaptation with respect to the λ shock at the upper wing surface and improved resolution at the leading edge.

Applications for 2D Navier-stokes Some adaptation results for 2D structured grids suitable for solution of the Navier-Stokes equations have been presented in [35], [36] [37], [38] and [29]. Luong, Thompson and Gatlin [35] show applications of the control function approach [17] for two low Reynolds number flows; a back-facing step and a turnaround duct. Klopfer [36] shows some impressive results for a hypersonic high Reynolds number flow over a cilinder with an impinging shock introducing shock-shock and shock-boundary layer interactions. Klopfer uses a hyperbolic grid adaptor which is possible because the grid away from the cilinder is free to move. Slater, Liou and Hindman [37] show a strong coupling between flow solver and grid adaptor resulting in dynamic adaptation. They apply their method, based on the Brackbill and Saltzman equations [24], to a supersonic high Reynolds number shock-boundary layer interaction. Finally Hall and Zing [38] present adaptive viscous airfoil computations using the control function approach formulated by Eiseman [3]. Compared to grid independent flow solutions they succeed to improve significantly on the accuracy of the aerodynamic coefficients. The results of Hagmeijer [29] for the RAE2822 airfoil are discussed in detail in Part2 of this lecture.

Applications for 3D Navier-stokes Some adaptation results for 3D structured grids suitable for solution of the Navier-Stokes equations have been presented in [39],[40],[41],[42] and [31]. Harvey, Acharya and Lawrence [39] use the tension and torsion spring analogy of Nakahashi and Deiwert [16] to the hypersonic high Reynolds-number flow calculation over a cone. Kania [40] shows results obtained with an 3D extension of the 2D method of Connett et. al. [14] for the supersonic high Reynolds-number flow over a blunted cone and the transonic high Reynolds-number flow over a fuselage forebody. Bockelie and Smith [41] show results for hypersonic high Reynolds-number flow over a re-entry body using a multi-phase alternate direction algorithm. Henderson, Huang, Lee and Choo [42] use the method developed by Lee and Loellbach [21] to adapt grids for the computation of high Reynolds-number flows over a blunt fin and over the ONERA M6 wing. The results of Hagmeijer and Kok [31] for the ONERA M6 wing are discussed in detail in Part2 of this lecture.

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