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Particle filtering for stochastic hybrid systems

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Summary

The standard way of applying particle filtering to hybrid systems is to make use of hybrid particles, where each particle consists of two components, one assuming Euclidean values, and the other assuming discrete mode values. This paper develops a novel particle filter for a discrete-time stochastic hybrid system. The novelty lies in the use of the exact Bayesian equations for the conditional mode probabilities given the observations. Therefore particles are needed for the Euclidean valued state component only. The novel particle filter is referred to as the Interacting Multiple Model (IMM) particle filter because it has a switching/interaction step which is of the same form as the switching/interaction step of the IMM algorithm. Through Monte Carlo simulations, it is shown that the IMM particle filter has significant advantage over the standard particle filter, in particular for situations where conditional switching rate or conditional mode probabilities have small values.

Contents

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1 Introduction

The Sampling Importance Resampling (SIR) based particle filtering approach of [1] has shown to form an elegant and general approach towards the numerical evaluation of Bayesian nonlinear filtering equations [2], [3]. Most convincing are the applications where established non-linear filtering approaches do not work at all, whereas particle filtering does, such as the track-before-detect particle filter of [4].

For nonlinear filtering of a hybrid stochastic process $\{x_t, \theta_t\}$, with x_t assuming values in $\mathbb{R}^n$, and θ_t assuming values in $\mathbb{M} = \{1,...,N\}$, the SIR particle filter uses particles from the hybrid state space $\mathbb{R}^n \times \mathbb{M}$. Successful applications of this hybrid state space version of the SIR

particle filter have been shown for target tracking [5], [6], signal processing [7] and failure monitoring and diagnosis [8].

In the hybrid state space version of the SIR particle filter, each particle has two components (x_t^j, θ_t^j), with x_t^j assuming an Euclidean value, and θ_t^j assuming a discrete mode value. This approach works well as long as the conditional mode probabilities and/or switching rates do not assume very small values. Otherwise, there may be very few (or zero) particles for one or more of the mode values, and then the empirical density spanned by all particles with such a mode value does not form an accurate approximation of the corresponding exact conditional density. A brute force approach to compensate for this is a sufficient increase of the number of particles.

A more elegant approach is to evaluate the exact probability function on $\mathbb{M}$ analytically, and to use per θ-condition, $\mathbb{R}^n$-valued particles only. This approach is elaborated in this paper for a discrete time stochastic hybrid system. The resulting novel particle filter has a switching/interaction step which is of the same form as the switching/interaction step of the Interacting Multiple Model (IMM) filter algorithm [9], [10]. For this reason, the novel particle filter is referred to as the IMM particle filter. Through Monte Carlo simulations, it is shown that the IMM particle filter performs better than the standard particle filter, in particular for situations where conditional switching rate and/or conditional mode probabilities have small values.

The paper is organized as follows. First, in section II, we develop the exact Bayesian filter equations for the discrete time stochastic hybrid system considered. Subsequently, in section III, we develop the IMMPF. Next, in section IV, we compare the IMMPF with the standard PF and the IMM through Monte Carlo simulation. In section V, we draw conclusions.

2 State-dependent mode switching

2.1 Problem formulation

In this section we develop the exact Bayesian filter for hybrid state estimation for a discrete-time stochastic hybrid system. We consider the following system of stochastic difference equations, on [0,T], T<∞,

$$x_t = a(\theta_t, x_{t-1}) + b(\theta_t, x_{t-1}) w_t \tag{1}$$
$$\theta_t = c(\theta_{t-1}, x_{t-1}, u_t) \tag{2}$$
$$y_t = h(\theta_t, x_t) + g(\theta_t, x_t) v_t \tag{3}$$

where $\{w_t\}$, $\{u_t\}$ and $\{v_t\}$ are independent sequences of i.i.d. standard Gaussian variables of dimension p, 1 and q respectively, the initial density-mass of (x_0,θ_0) is p_{x_0,θ_0}, and $\{w_t, v_t, u_t\}$ is independent of (x_0,θ_0). Furthermore x_t,θ_t and y_t have respectively $\mathbb{R}^n$ -, $\mathbb{M}$ - and $\mathbb{R}^m$ - valued realizations, with $\mathbb{M} = \{1,...,N\}$, while *a, b, c, h* and *g* are measurable mappings of appropriate dimensions such that system (1-3) has a unique solution for each initial (x_0,θ_0). The mappings *a, b, c, h* and *g* are time-invariant for notational simplicity only.

In this model, the pair (x_t,θ_t) represents the hybrid system state, while y_t represents the measurement. Since equations (1) and (2) are not commonly used as a hybrid state model, we give a short introduction to them, starting with (2).

Since $\{\theta_t\}$ assumes values in a discrete set $\mathbb{M}$, while *c* is a mapping of $\mathbb{M}\times\mathbb{R}^n\times\mathbb{R}$ into $\mathbb{M}$, equation (2) induces state-dependent mode transition probabilities $\Pi_{\eta\theta}(x)$ as follows:

$$\begin{aligned}\Pi_{\eta\theta}(x) &\triangleq p_{\theta_t|\theta_{t-1},x_{t-1}}(\theta\,|\,\eta,x) = \\ &= \int_{\mathbb{R}} \chi\big(\theta, c(\eta,x,u)\big)\, p_{u_t}(u)du\end{aligned} \tag{4}$$

with χ a 0-1 indicator; $\chi(\theta,\eta)$= 1 iff $\theta=\eta$. From (1) and (4) it follows that $\{\theta_t, x_t\}$ is a hybrid state Markov proces.

2.2 Bayesian filter equations

Bayesian filtering asks for recursive equations for the evolution of the conditional density-mass function $p_{x_t,\theta_t|Y_t}$, with $Y_t = \{y_s; s \le t\}$. To develop such equations, we decompose a filter cycle into a sequence of transitions:

$$p_{\theta_{t-1}|Y_{t-1}} \xrightarrow{\text{I.a}} p_{\theta_t|Y_{t-1}}$$
$$p_{x_{t-1}|\theta_{t-1},Y_{t-1}} \xrightarrow{\text{I.b}} p_{x_{t-1}|\theta_t,Y_{t-1}}$$

$$
\begin{array}{lcl}
p_{x_{t-1}|\theta_t,Y_{t-1}} & \xrightarrow{\text{I.c}} & p_{x_t|\theta_t,Y_{t-1}} \\
p_{\theta_t|Y_{t-1}} & \xrightarrow{\text{I.d}} & p_{\theta_t|Y_t} \\
p_{x_t|\theta_t,Y_{t-1}} & \xrightarrow{\text{I.e}} & p_{x_t|\theta_t,Y_t}
\end{array} \tag{5}
$$

which may be combined for every *t*, *s* and *u* through

$$p_{x_t,\theta_s|Y_u}(\cdot,\theta) = p_{x_t|\theta_s,Y_u}(\cdot,\theta)\, p_{\theta_s|Y_u}(\theta)$$

Transitions I.c-e in (5) are known from the Markov switching and non-switching situations. Characterizations for I.a,b in (5) are given in the Theorem below.

Theorem:
Of the sequence of elementary transitions in (5) the first two transitions satisfy:

$$p_{\theta_t|Y_{t-1}}(\theta) = \sum_{\eta\in\mathbb{M}} \hat{\Pi}_{t-1,\eta\theta}\, p_{\theta_{t-1}|Y_{t-1}}(\eta) \tag{I.a}$$

with:

$$\hat{\Pi}_{t-1,\eta\theta} \triangleq p_{\theta_t|\theta_{t-1},Y_{t-1}}(\theta\,|\,\eta) = \int_{\mathbb{R}^n} p_{x_{t-1}|\theta_{t-1},Y_{t-1}}(x\,|\,\eta)\Pi_{\eta\theta}(x)dx$$

and if $p_{\theta_t|Y_{t-1}}(\theta) > 0$,

$$p_{x_{t-1}|\theta_t,Y_{t-1}}(\cdot\,|\,\theta) = \sum_{\eta\in\mathbb{M}} \left(p_{x_{t-1}|\theta_{t-1},Y_{t-1}}(\cdot\,|\,\eta)\cdot\Pi_{\eta\theta}(\cdot)\, p_{\theta_{t-1}|Y_{t-1}}(\eta)\right) / \, p_{\theta_t|Y_{t-1}}(\theta) \tag{I.b}$$

Proof: Since θ_t is conditionally independent of Y_{t-1} given θ_{t-1} and x_{t-1}, we get

$$
\begin{aligned}
p_{\theta_t|\theta_{t-1},Y_{t-1}}(\theta\,|\,\eta) &= \int_{\mathbb{R}^n} p_{x_{t-1}|\theta_{t-1},Y_{t-1}}(x\,|\,\eta)\, p_{\theta_t|\theta_{t-1},x_{t-1}}(\theta\,|\,\eta,x)dx = \\
&= \int_{\mathbb{R}^n} p_{x_{t-1}|\theta_{t-1},Y_{t-1}}(x\,|\,\eta)\Pi_{\theta\eta}(x)dx = \hat{\Pi}_{t-1,\eta\theta}
\end{aligned}
$$

From this (I.a) follows directly. For (I.b) see the Appendix.

Remark: If the condition $p_{\theta_t|Y_{t-1}}(\theta) > 0$ is not satisfied, then $p_{x_{t-1},\theta_t|Y_{t-1}}(\cdot,\theta)$ in eq. (A.2) of Appendix A characterizes an unnormalized version of $p_{x_{t-1}|\theta_t,Y_{t-1}}(\cdot\,|\,\theta)$.

Obviously, the non-Markovian character of the switching mode process $\{\theta_t\}$ is reflected by the appearance of $\hat{\Pi}_{t,\eta\theta}$ in (I.a) and of $\Pi_{\eta\theta}(\cdot)$ in (I.b). If *b*, *c* and *g* are *x*-invariant and *a* and *h* are linear in *x*, then $\{\theta_t\}$ is a Markov process and (1,3) is jump linear. In this case $p_{x_t|\theta_t,Y_t}(.|\theta)$ is a mixture of N^{t+1} Gaussian densities [11]. Because of the appearance of the term $\Pi_{\eta\theta}(\cdot)$ in (I.b), however, this does not hold true for the state-dependent switching case.

3 IMM Particle Filter

One cycle of the IMM Particle Filter consists of the following five steps, where a particle is defined as a pair (μ, x), $\mu \in [0,1]$, and $x \in \mathbb{R}^n$.

IMM Particle Filter Step 1; Each filter starts per mode value with a set of $S' = S/N$ particles in $[0,1] \times \mathbb{R}^n$:

$$\left\{\left(\mu_t^{\theta,j}, x_t^{\theta,j}\right); j \in [1, S']\right\},\ \theta \in \mathbb{M} = \{1,...,N\}$$

with for $t = 0$, $\mu_0^{\theta,j} = p_{\theta_0}(\theta)/S'$ and $x_0^{\theta,j}$ independently drawn from $p_{x_0|\theta_0}(\cdot\,|\,\theta)$.

IMM Particle Filter Step 2; Interaction based re-sampling:
Based on eq. (I.a), the mode probabilities become:

$$p_{\theta_{t+1}|Y_t}(\theta) \approx \overline{\gamma}_{t+1}(\theta) = \sum_{\eta} \sum_{j=1}^{S'} \Pi_{\eta\theta}(x_t^{\eta,j}) \mu_t^{\eta,j}$$

For each $\theta \in \mathbb{M}$, draw S' random vectors $\overline{x}_t^{\theta,j}$, $j \in [1, S']$, from the particle spanned density in eq. (I.b):

$$p_{x_t|\theta_{t+1},Y_t}(\cdot\,|\,\theta) \approx \sum_{\eta \in \mathbb{M}} \sum_{j=1}^{S'} \left(\Pi_{\eta\theta}(x_t^{\eta,j}) \mu_t^{\eta,j} \delta_{x_t^{\eta,j}}(\cdot)\right) / \overline{\gamma}_{t+1}(\theta)$$

or, if $\overline{\gamma}_t(\theta) = 0$, draw from an unnormalized version, e.g. the joint density in eq. (A.2):

$$p_{x_t,\theta_{t+1}|Y_t}(\cdot, \theta) \approx \sum_{\eta \in \mathbb{M}} \sum_{j=1}^{S'} \left(\Pi_{\eta\theta}(x_t^{\eta,j}) \mu_t^{\eta,j} \delta_{x_t^{\eta,j}}(\cdot)\right)$$

This yields for each $\theta \in \mathbb{M}$ the following set of particles

$$\left\{\left(\overline{\mu}_{t+1}^{\theta,j}, \overline{x}_t^{\theta,j}\right); j \in [1, S']\right\},\ \theta \in \mathbb{M}$$

with $\overline{\mu}_{t+1}^{\theta,j} = \overline{\gamma}_{t+1}(\theta)/S'$ for any *j*.

IMM Particle Filter Step 3; Determine the new set of particles (the weights are not changed)

$$\left\{\left(\bar{\mu}_{t+1}^{\theta,j}, x_{t+1}^{\theta,j}\right); j \in [1, S']\right\}, \ \theta \in \mathbb{M}$$

by running for each particle a Monte Carlo simulation from t to $t+1$ according to eq. (1):

$$x_{t+1}^{\theta,j} = a\left(\theta, \bar{x}_t^{\theta,j}\right) + b(\theta, \bar{x}_t^{\theta,j}) w_{t+1}^j$$

IMM Particle Filter Step 4: Measurement update of the new weights for the set of particles, i.e.:

$$\left\{\left(\mu_{t+1}^{\theta,j}, x_{t+1}^{\theta,j}\right); j \in [1, S']\right\}, \ \theta \in \mathbb{M}$$

with for the new weights

$$\mu_{t+1}^{\theta,j} = \bar{\mu}_{t+1}^{\theta,j} \cdot \frac{1}{c_{t+1}} F_{t+1}(x_{t+1}^{\theta,j}, \theta)$$

where

$$F_{t+1}(x,\theta) = \left((2\pi)^m \operatorname{Det}\left\{\tilde{Q}_{t+1}(x,\theta)\right\}\right)^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}\tilde{v}_{t+1}^T(x,\theta)\tilde{Q}_{t+1}(x,\theta)^{-1}\tilde{v}_{t+1}(x,\theta)\right\}$$

with

$$\tilde{v}_{t+1}(x,\theta) \triangleq y_{t+1} - h(\theta, x)$$
$$\tilde{Q}_{t+1}(x,\theta) \triangleq g(\theta,x) g(\theta,x)^T$$

and c_{t+1} such that

$$\sum_{\theta \in \mathbb{M}} \sum_{j=1}^{S'} \mu_{t+1}^{\theta,j} = 1$$

IMM Particle Filter Step 5: MMSE output equations:

$$\hat{\gamma}_{t+1}(\theta) = \sum_{j=1}^{S'} \mu_{t+1}^{\theta,j}$$

$$\hat{\gamma}_{t+1}(\theta)\hat{x}_{t+1}(\theta) = \sum_{j=1}^{S'} \mu_{t+1}^{\theta,j} x_{t+1}^{\theta,j}$$

$$\hat{\gamma}_{t+1}(\theta)\hat{P}_{t+1}(\theta) = \sum_{j=1}^{S'} \mu_{t+1}^{\theta,j} \left[x_{t+1}^{\theta,j} - \hat{x}_{t+1}(\theta) \right] \left[x_{t+1}^{\theta,j} - \hat{x}_{t+1}(\theta) \right]^T$$

$$\hat{x}_{t+1} = \sum_{\theta \in \mathbb{M}} \hat{\gamma}_{t+1}(\theta)\hat{x}_{t+1}(\theta)$$

$$\hat{P}_{t+1} = \sum_{\theta \in \mathbb{M}} \hat{\gamma}_{t+1}(\theta) \left(\hat{P}_{t+1}(\theta) + [x_{t+1}(\theta) - \hat{x}_{t+1}][x_{t+1}(\theta) - \hat{x}_{t+1}]^T \right)$$

Notice the differences with the standard PF for hybrid systems [5], [7], [8]:

- Fixed number of particles per mode.
- Probabilities for $\{\theta_t\}$ instead of particles for $\{\theta_t\}$,
- Resampling after interaction/mixing rather than after measurement update.

4 Monte Carlo simulations

In this section some Monte Carlo simulation results are given for the IMM Particle Filter (IMMPF), the standard Particle Filter (PF) and the IMM algorithm. In addition we also give simulation results for a Hybrid Particle Filter (HPF) which differs from the standard PF by using a fixed number of particles per mode as introduced by [12]. For each of the particle filters we used a total of S = 10000 and S = 1000 particles respectively. The simulations primarily aim at gaining insight in the behavior and performance of the filters in case of rare switching. In the example scenarios there is an object moving with two possible modes. One mode is constant velocity and the other mode is constant acceleration. The object starts with zero velocity and continues this for 40 scans. After scan 40 the object starts to accelerate with at a value equal to the standard deviation σ_a of acceleration values. In scenarios 1 and 2 the object continues with constant velocity after scan 60, while in scenarios 3 and 4 the object continues accelerating. Each simulation the filters start with perfect estimates and run for 100 scans. The hybrid model considered is a Markovian jump linear system:

$$x_t = A(\theta_t)x_{t-1} + B(\theta_t)w_t \tag{6}$$

$$y_t = H(\theta_t)x_t + G(\theta_t)v_t \tag{7}$$

with parameterization $\theta_t \in \{1,2\}$ and

$$A(1) = \begin{bmatrix} 1 & T_s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 1 & T_s & \frac{1}{2}T_s^2 \\ 0 & 1 & T_s \\ 0 & 0 & \alpha \end{bmatrix}, \quad B(1) = \sigma_a \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B(2) = \sigma_a \begin{bmatrix} 0 \\ 0 \\ \sqrt{1-\alpha^2} \end{bmatrix}$$

$$H(\cdot) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad G(\cdot) = \sigma_m$$

$$\Pi = \begin{bmatrix} 1-\frac{T_s}{\tau_1} & \frac{T_s}{\tau_1} \\ \frac{T_s}{\tau_2} & 1-\frac{T_s}{\tau_2} \end{bmatrix}$$

where σ_a represents the standard deviation of acceleration noise and σ_m represents the standard deviation of the measurement error. Table I gives the scenario parameter values that are being used for the Monte Carlo simulations.

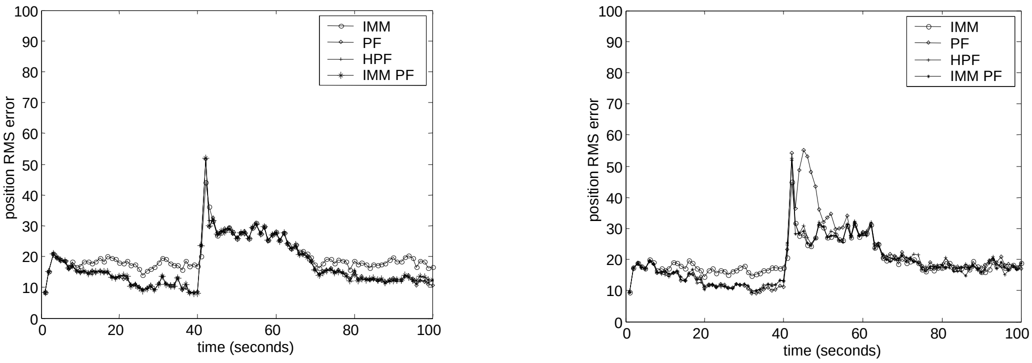
Table I. Scenario parameter values

Scenario	α	σ_a	τ_1	τ_2	T_S
1	0.9	50	50	5	1
2	0.9	50	5000	5	1
3	0.9	1	50	5	1
4	0.9	1	5000	500	1

For each of the scenarios Monte Carlo simulations containing 100 runs have been performed for each of the filters. To make the comparison more meaningful, for all filters the same random number streams were used. The results of the Monte Carlo simulations of the four scenarios are shown in figures are shown in tables and figures as follows:

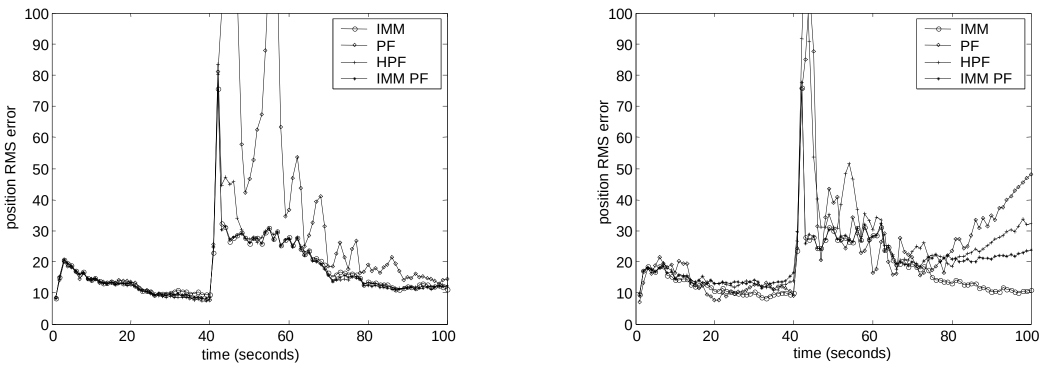
The position RMS errors in figures 1,2,3 and 4.

The computational load in Table II.



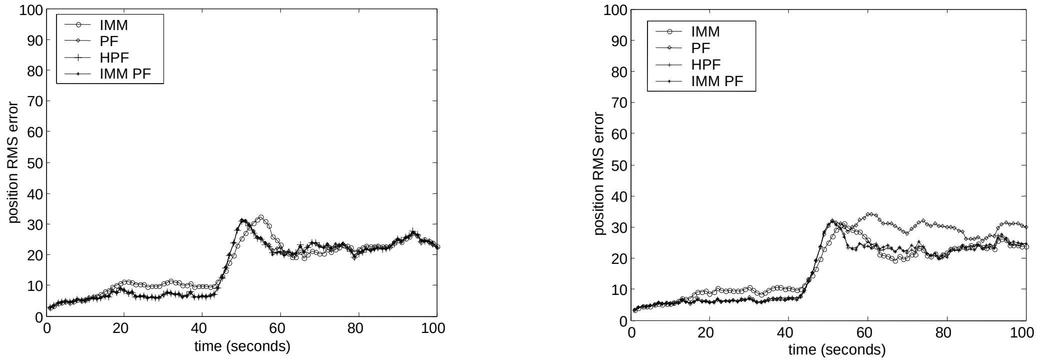
a. 10^4 particles *b. 10^3 particles*

Figure 1. Scenario 1



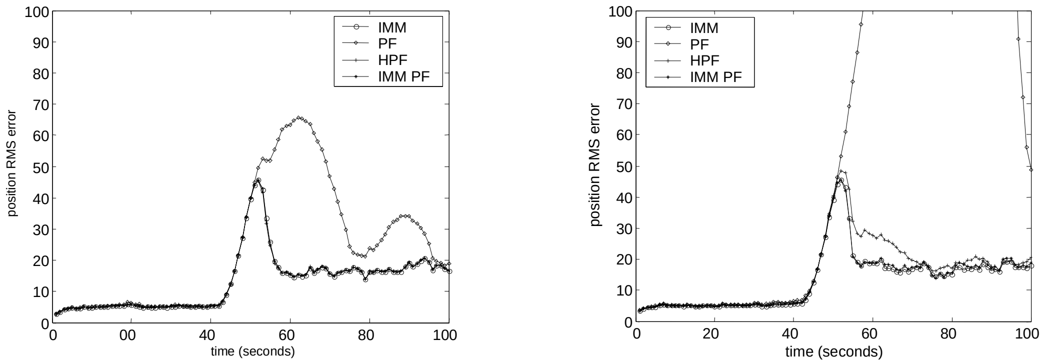
a. 10^4 particles *b. 10^3 particles*

Figure 2. Scenario 2



a. 10^4 particles *b. 10^3 particles*

Figure 3. Scenario 3



a. 10^4 particles *b. 10^3 particles*

Figure 4. Scenario 4

Table II. Computational load per scan (10^{-3} s)

# particles	IMM	PF	HPF	IMMPF
10^4	4	138	115	96
10^3	4	19	13	11

<u>Scenario 1:</u>

With 10^4 particles, all three particle filters perform similarly well; they converge to a lower value during uniform motion than IMM does. As a side effect, the peak RMS error at the start of acceleration is for the particle filters slightly higher than it is for IMM. These results agree well with those in [6]. Reduction of the number of particles to 10^3 affects PF dramatically, but has negligible impact on HPF and IMMPF.

Scenario 2:

With 10^4 particles, IMMPF performs marginally better than IMM does, while PF performs dramatically worse. HPF performs significantly worse during the initial acceleration period only. Reduction of the number of particles to 10^3 has a negative effect on the convergence during UM for all three particle filters. Moreover, during the period of acceleration, PF and HPF worsen dramatically in performance.

Scenario 3:

With 10^4 particles, all three particle filters perform equally well, and significantly better than IMM does. Reduction of the number of particles to 10^3 has a clear negative effect for the standard PF, but does not affect IMMPF and HPF.

Scenario 4:

With 10^4 particles, all four filters, except the standard PF, perform similarly well. The standard PF performs dramatically worse during CA. Reduction of the number of particles to 10^3 has a clear negative effect for the standard PF and the HPF, but not for the IMMPF.

Summary of Monte Carlo simulation results:

With 10^4 particles, all three particle filters perform better than IMM for scenarios 1 and 3. For scenarios 2 and 4 however, IMM and IMMPF perform similarly well, while the standard PF performs less good on sudden acceleration, and the HPF response is less good for acceleration in scenario 2 only. With 10^3 particles the performance of PF degrades for all scenarios, of HPF for scenarios 2 and 4, and of IMMPF for scenario 2 only.

5 Concluding remarks

In this paper we developed a novel particle filter for discrete time stochastic hybrid systems. Because of its similarity with the switching/interaction step of IMM, this novel particle filter is referred to as IMM Particle Filter. Through MC simulations for four scenarios; IMMPF has been tested and compared with standard PF and IMM.

With 10^4 particles, the IMMPF performs well for all four scenarios. The computational load is 25 times the load of IMM. The computational load of the standard PF is even higher. As expected, the IMMPF works well for all four scenarios including ones where standard PF or IMM has problems. Hence IMMPF is the preferred particle filter for stochastic hybrid systems. For the scenarios with a small switching rate (scenarios 2 and 4), the IMM performs similarly well as the IMMPF.

However, for the regular switching scenarios 1 and 3, the IMMPF has some performance advantage over IMM, also when the number of particles is down to 10^3. The computational load of IMMPF is then three times higher than the load of IMM. Because IMMPF can easily be combined with various kinds of deviations from the Markovian jump linear mode (i.e. non-linear a or h, or x-dependent b, c or g) this means that IMMPF is a strong competitor of IMM. Follow-up research is to combine IMMPF with complementary methods (e.g. [13]) to mitigate sensitivity to divergence for scenario 2 in case of 10^3 particles, and to gain analytical insight in convergence characteristics.

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Appendix A Appendix A

In this appendix we derive characterizations of the elementary transitions I.b in the Theorem. From the law of total probability follows:

$$\begin{aligned} p_{x_{t-1},\theta_t|Y_{t-1}}(x,\theta) &= \sum_{\eta} p_{x_{t-1},\theta_t,\theta_{t-1}|Y_{t-1}}(x,\theta,\eta) = \\ &= \sum_{\eta} p_{\theta_t|x_{t-1},\theta_{t-1},Y_{t-1}}(\theta\,|\,x,\eta)\, p_{x_{t-1},\theta_{t-1}|Y_{t-1}}(x,\eta) = \\ &= \sum_{\eta} \Pi_{\eta\theta}(x)\, p_{x_{t-1},\theta_{t-1}|Y_{t-1}}(x,\eta) \end{aligned} \tag{A.8}$$

Hence,

$$p_{x_{t-1},\theta_t|Y_{t-1}}(x,\theta) = \sum_{\eta} \Pi_{\eta\theta}(x)\, p_{x_{t-1}|\theta_{t-1},Y_{t-1}}(x\,|\,\eta)\, p_{\theta_{t-1}|Y_{t-1}}(\eta) \tag{A.9}$$

If $p_{\theta_t|Y_{t-1}}(\theta) > 0$ for all $\theta \in \mathbb{M}$, then (A.2) yields (I.b).