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Summary

Many aerospace systems with operational and environmental restrictions are examples of constrained dynamical systems. Mathematical models of constrained dynamical systems usually consist of a combination of equalities and inequalities. In this report some examples are presented to illustrate this fact.

Some of the results obtained and difficulties encountered in our research towards a generalisation of a result first derived by Farkas in 1895 are presented. Such a novel Farkas Lemma is necessary in order to arrive at a systematic approach to deriving efficient representations of constrained dynamical systems.

Contents

(16 pages in total)

1 Introduction

The National Aerospace Laboratory NLR is actively involved in engineering of aerospace systems such as aircraft, satellites and robotic manipulators. Aerospace systems are complex systems that contain many subsystems, each with its own specific function. More often than not, aerospace systems are designed in international project teams, where companies are responsible for specific subsystems. Consequently, system integration aspects also involve the specification and verification of (sub)system interactions, and operational and environmental restrictions. Dynamical systems that include (descriptions of) restrictions are often referred to as constrained dynamical systems.

An important aspect of aerospace systems is that in many cases the environment in which a final product will operate is not readily accessible. This fact is obvious (at least today) for systems that will operate in space, such as a robotic manipulator. Simulation studies offer the opportunity to gain experience with the system to be designed already at an early stage of a project. In addition, simulation allows for early testing of various interactions between subsystems, and for early testing of the control laws that are being designed to direct the system. In case of digital simulation, the need for mathematical models and methods that are capable of handling system complexity and interactions is apparent.

Modelling operational and environmental restrictions of systems often leads to inequalities. For systems described by equalities many algorithms are available to arrive at efficient representations, e.g. representations that contain no redundant equations, however, similar results for systems that involve inequalities are not so abundant. For static inequality systems, a constructive algorithm to obtain minimal representations can be found in [7,11(chapter 3)]. The extension of these results to dynamic inequality systems is the main subject of this report.

In chapter 2 we will present a number of examples to illustrate that modelling of real-world systems with real-world restrictions leads in many cases to inequalities in the mathematical model. Chapter 3 contains a contribution to the book 'Open Problems in Mathematical Systems and Control Theory'. Some of the results obtained and difficulties encountered in our research towards a generalisation of a result first derived by Farkas in 1895 are presented. Concluding remarks can be found in chapter 4. Appendix A gives an overview of the notation that is used in this report.

2 Illustrative Examples

In applied mathematics, inequalities have since long been used to describe certain physical phenomena. Fourier discussed inequalities motivated by the observation that constraints in mechanics are often 'one-sided'. This led him to formulate what is now known as the mechanical principle of Fourier, which states that for a solid body in equilibrium the sum of the moments of the applied forces satisfies an inequality [16]. At the end of the 19th century Farkas studied static linear inequalities. Farkas' interest in linear inequalities can be traced back to his interest in Fourier's principle. A result obtained by Farkas in 1895, and known as Farkas Lemma [13], was used by Kuhn and Tucker in 1951 to derive necessary conditions for optimality for nonlinear programming theory, leading to a rapid development of nonlinear optimization theory [16].

Most research on dynamical systems where part of the behaviour can be described by inequalities began after the advent of game theory and mathematical programming. This was motivated principally by questions in economics and operations research. In system theory, inequalities are also discussed, but here attention was until recently almost exclusively restricted to systems in first-order form. Many applications however, yield models that are not in first-order form. Moreover, in many applications inequalities appear as a natural part in the description of the system. In the remainder of this chapter we will give a series of examples ranging from elementary systems to complex physical systems. First we give an example where the model is given by inequalities only.

Example 2.1 *A Leontief economy* [14]. Consider an economy that produces at each stage of the production cycle amounts x_i, $i \in \{1, \ldots, n\}$, of n products. Assume that in order to produce one new item of the jth product one needs at least a_{ij} units of product i. The a_{ij}'s are called the technology coefficients. This induces the following constraints on the feasible production paths $\{x(t), t \in \mathbb{Z}_+\}$:

$$x_i(t) \geq \sum_{j=1}^{n} a_{ij} x_j(t+1), \ \forall t \in \mathbb{Z}_+, \tag{1}$$

with the natural constraint

$$x_i(t) \geq 0, \ \forall t \in \mathbb{Z}_+. \tag{2}$$

The difference between the left and right side of inequality (1) will be due to, for instance, imbalance of the available products and consumption. $\triangleleft$

Also in the field of electrical systems, inequalities play an important role.

Example 2.2 *An ideal diode.* An ideal diode is an electrical device which allows currents to flow in one direction but blocks it from flowing in the other direction.

Let V denote the voltage and I denote the current. Then

$$\begin{aligned} I \geq 0 \quad &\wedge \quad V = 0, \text{ or} \\ V \leq 0 \quad &\wedge \quad I = 0. \end{aligned} \tag{3}$$

The system can also be represented by the following conditions:

$$\begin{aligned} I &\geq 0, \text{ and} \\ V &\leq 0, \text{ and} \\ V \cdot I &= 0. \end{aligned} \tag{4}$$

Note that now part of the behaviour is represented by a nonlinear equality. Of course, (models for) electrical circuits will contain many diodes and other devices. $\triangleleft$

In the field of mechanical systems research into the mechanics of contact, i.e. interaction of a system with obstacles in its environment has a long history [1, 2]. A simple example is given next.

Example 2.3 *A falling ball and a circular basin* [5]. Consider a ball of mass m that is constrained to move in a circular basin of radius r. Let the position of the ball be denoted by $(y_1, y_2)^T \in \mathbb{R}^2$. Let g denote the constant of gravity.

A mathematical model of the position of the ball is given by

$$\begin{aligned} m \, \frac{d^2 y_1}{dt^2}(t) &= 0, \\ m \, \frac{d^2 y_2}{dt^2}(t) &= -m \, \mathrm{g}, \end{aligned} \tag{5}$$

for all $t \in \mathbb{R}_+$. The motion of the ball is on the boundary of the basin, which gives

$$\begin{aligned} y_1^2(t) + y_2^2(t) &= r^2, \\ y_2(t) &\leq 0, \end{aligned} \tag{6}$$

for all $t \in \mathbb{R}_+$. It is possible to give a dynamics model without constraint equations if we assume that $y_2(0) \leq 0$ holds initially. Define the angle ϕ such that

$$\begin{aligned} y_1 &= r \, \sin(\phi), \\ y_2 &= -r \, \cos(\phi). \end{aligned} \tag{7}$$

An alternative dynamics model can now be given as

$$m\,\frac{d^2\phi}{dt^2}(t) + \frac{m\,\mathrm{g}}{r}\,\sin(\phi(t)) = 0,\ \forall t \in \mathbb{R}_+. \tag{8}$$

This model can also be interpreted as a description of an undamped pendulum. Note that we still need the kinematic equations (7) of the system to obtain the trajectories in terms of the original variables.

Next consider the same ball, but now assume that the ball is initially above the basin. In this case the first equation in (6) must be replaced by

$$y_1^2(t) + y_2^2(t) \le r^2. \tag{9}$$

This inequality is not so easily eliminated from the governing equations. $\triangleleft$

It can be seen that further detailing of a model or modelling additional features (or requirements) of a system at hand, may result in a new model that again has some of the original model characteristics, such as the presence of inequalities, despite the earlier effort to eliminate some of these characteristics. As a consequence, it may be advantageous to maintain the original (sub)models and to treat further interaction constraints simply as additional equations. This implies that, as a default, system representations contain both equalities and inequalities.

The following two examples are taken from NLR's involvement in the design of aerospace systems.

Example 2.4 *Constrained robotic manipulator* [2, 9]. A robotic system consists of a series of links, where two neighbouring links are connected by joints. A model for a rigid manipulator without gearbox flexibility is given by

$$M(q(t))\frac{d^2q}{dt^2}(t) + N(q(t), \frac{dq}{dt}(t)) = \tau,\ \forall t \in \mathbb{R}_+. \tag{10}$$

Here q denotes the n-vector of joint angles, $M(q)$ denotes the inertia matrix function, while $N(q, \frac{dq}{dt})$ is a vector function which characterizes the Coriolis, centrifugal and gravitational load of the manipulator, and τ is the vector of inputs. Assume the following relation between end-effector position y, and velocity $\frac{dy}{dt}$, and joint values q, and joint angular values $\frac{dq}{dt}$, respectively:

$$\begin{aligned} y(t) &= H(q(t)), && (11)\\ \frac{dy}{dt}(t) &= J(q(t))\frac{dq}{dt}(t), && (12) \end{aligned}$$

for all $t \in \mathbb{R}_+$. Here $J(:= \frac{\partial H}{\partial q})$ is called the manipulator Jacobian matrix. In this setting the differential equations model the unconstrained behaviour of the manipulator. Inevitably, during operations with a robotic arm situations will occur where the manipulator is, or comes, in contact with its environment. The environmental restrictions can be modelled by algebraic equalities and inequalities. For instance, a manipulator that is in contact with a satellite is not allowed to damage this satellite. This clearly puts restrictions on the possible control choices. If we model the satellite simply as a surface in the environment this gives rise to a unilateral constraint, i.e. it is assumed that the constraint surface can be approached from one side only. This gives

$$\phi(y(t)) \geq 0, \ \forall t \in \mathbb{R}_+, \tag{13}$$

where the boundary of this unilateral constraint models the surface. Interaction with the surface will inevitably mean restrictions on the position, velocity and forces. In addition, there are usually physical limitations on joint rotations

$$\alpha \leq q_i(t) \leq \beta, \ i \in \underline{n}, \ \forall t \in \mathbb{R}_+, \tag{14}$$

which imply further restrictions on the behaviour. $\triangleleft$

Example 2.5 *Wind tunnel experiments* [4]. During a test in a wind tunnel, a model of an aerospace object moves within the test section of this tunnel. The model is mounted on a special type of robotic manipulator. The test section is modelled as a rectangular box, with virtual walls in front and back of the model to ensure optimal measurement conditions. Let (x, y, z) denote a position in a Cartesian coordinate system. The region of the test section in which tests can be conducted in a safe manner can be modelled as:

$$\begin{aligned} x_{min} + \delta \leq \quad x \quad &\leq x_{max} - \delta, \\ y_{min} + \delta \leq \quad y \quad &\leq y_{max} - \delta, \\ z_{min} + \delta \leq \quad z \quad &\leq z_{max} - \delta. \end{aligned} \tag{15}$$

The parameter δ ($\delta > 0$) is a safety parameter. The problem at hand is the following. During the moves, collisions between manipulator or model on the one hand and the test section walls on the other hand must be avoided. For this, a number of so called critical points are chosen on the manipulator and the model of the aerospace object. For instance, the tip of a wing of an aeroplane is usually taken as a critical point. Suppose that there are k critical points, denoted by (x_i, y_i, z_i), $i \in \underline{k}$. The problem is to plan and control the moves such that all points (x_i, y_i, z_i) satisfy (15) during the complete move. $\triangleleft$

It is worth emphasizing that although examples 2.4 and 2.5 both deal with a robotic manipulator that is restricted in its behaviour, the operational goals are quite different; the robotic manipulator in example 2.4 is deliberately put in contact with its environment, whereas the robotic manipulator in example 2.5 should avoid contact with its environment.

We end this chapter with an example taken from studies on vehicle behaviours (see also [15]).

Example 2.6 *Carts riding on a track* (modified from [10]). Suppose that two carts are moving on the same track, where the second cart is initially to the right of the first cart. Denote y_1, y_2, the position of the first cart, the second cart, respectively. Let u_1 and u_2 denote the control vectors.

The dynamics equations are given by:

$$M_1(y_1(t))\frac{d^2y_1}{dt^2}(t) + N_1(y_1(t), \frac{dy_1}{dt}(t)) \quad = \quad u_1(t), \tag{16}$$
$$M_2(y_2(t))\frac{d^2y_2}{dt^2}(t) + N_2(y_2(t), \frac{dy_2}{dt}(t)) \quad = \quad u_2(t), \tag{17}$$

for all $t \in \mathbb{R}_+$. The requirement that the second cart must remain to the right of the first cart reads:

$$y_2(t) - y_1(t) \geq 0, \ \forall t \in \mathbb{R}_+. \tag{18}$$

If $y_2(t) = y_1(t)$ for some $t \in \mathbb{R}_+$ then the carts are in contact with each other. $\triangleleft$

Other examples of mathematical models that involve inequalities can be found in for instance flight path planning of aircraft, and modelling of the behaviour of a condenser for a two-phase heat transport system [3, 8].

Constrained dynamical systems, of which examples have been presented in this chapter will be discussed in a general mathematical framework in the next chapter, where we pursue a generalisation of Farkas Lemma to so called behavioral inequalities. Such a generalisation is crucial to the development of algorithms to obtain models of restricted dynamical systems. The research will be done in a discrete-time linear setting. (Recall that application of numerical methods to continuous-time systems also yield discrete-time systems.) The straightforward process of rewriting models obtained from first principles into behavioral (in)equalities can be found in [11, 18]. Note that any equality can always be written as a system of two inequalities.

3 A Farkas Lemma for Behavioral Inequalities

3.1 Description of the problem

Within the systems and control community there has always been an interest in minimality issues. In this chapter[1] we conjecture a Farkas Lemma for behavioral inequalities that, when true, will allow to study minimality and elimination issues for behavioral systems described by inequalities.

Let $\mathbb{R}^{n\times m}[s,s^{-1}]$ denote the $(n\times m)$ polynomial matrices with real coefficients and positive and negative powers in the indeterminate s. Let $\mathbb{R}_+^{n\times m}[s,s^{-1}]$ denote the set of matrices in $\mathbb{R}^{n\times m}[s,s^{-1}]$ with nonnegative coefficients only. In this chapter we consider discrete-time systems with time-axis $\mathbb{Z}$. Let σ denote the (backward) shift operator, and let $R(\sigma,\sigma^{-1})$ denote polynomial operators in the shift.

Of interest is the relation between two polynomial matrices $R(s,s^{-1})$ and $R'(s,s^{-1})$ when they satisfy

$$R(\sigma,\sigma^{-1})w \geq 0 \Rightarrow R'(\sigma,\sigma^{-1})w \geq 0. \tag{19}$$

Based on the static case, one may expect that such a relation should be the extension of Farkas Lemma to the behavioral case. This leads to the raison d'être of this chapter.

Conjecture 3.1 *Let* $R \in \mathbb{R}^{g\times q}[s,s^{-1}]$ *and* $R' \in \mathbb{R}^{g'\times q}[s,s^{-1}]$. *Then:* $\{R(\sigma,\sigma^{-1})w \geq 0 \Rightarrow R'(\sigma,\sigma^{-1})w \geq 0\}$ *if and only if there exists a polynomial matrix* $H \in \mathbb{R}_+^{g'\times g}[s,s^{-1}]$ *such that* $R'(s,s^{-1}) = H(s,s^{-1})R(s,s^{-1})$. $\lhd$

In order to prove this conjecture, one could try to extend the original proof given by Farkas in [13]. However, this proof explicitly uses the fact that every scalar that is unequal to zero is invertible. Such a general statement does not hold for elements of $\mathbb{R}^{g\times q}[s,s^{-1}]$. The most promising approach for the dynamic case seems to be the use of mathematical tools such as the separation theorem of Hahn-Banach (see for instance [17]). The basic mathematical preliminaries read as follows.

Denote $\mathbb{E} := (\mathbb{R}^q)^{\mathbb{Z}}$ with the topology of point-wise convergence. The dual of $\mathbb{E}$, denoted by $\mathbb{E}^*$, consists of all $\mathbb{R}^q$-valued sequences that have compact support. Let $R \in \mathbb{R}^{g\times q}[s,s^{-1}]$. Let

[1] The content of this chapter has been submitted for inclusion in the book 'Open Problems in Mathematical Systems and Control Theory', authors A.A. ten Dam and J.W. Nieuwenhuis (University Groningen)

$\mathfrak{B} = \{w \in \mathbb{E} | R(\sigma, \sigma^{-1})w \geq 0\}$. The polar cone of $\mathfrak{B}$, denoted by $\mathfrak{B}^{\#}$, is given by $\{w \in \mathbb{E}^* | \forall w \in \mathfrak{B} : \sum_{t \in \mathbb{Z}} w^*(t)w(t) \geq 0\}$. We would like to establish that $\mathfrak{B}^{\#} = \{w \in \mathbb{E} | \exists \alpha \in \mathbb{E}^*, \alpha \geq 0$ such that $w^* = R^T(\sigma^{-1}, \sigma)\alpha\}$, but we have so far not been able to prove or disprove these statements. These statements, together with the fact that $\{\mathfrak{B}_1 \subseteq \mathfrak{B}_2\}$ implies $\{\mathfrak{B}_2^{\#} \subseteq \mathfrak{B}_1^{\#}\}$ are believed to be useful in a proof of the conjecture.

3.2 Motivation and history of the problem

In the early nineties we started to investigate minimality issues for so called behavior inequality systems, e.g. systems whose behavior $\mathfrak{B}$ allows a description $\mathfrak{B} = \{w \in \mathbb{E} | R(\sigma, \sigma^{-1})w \geq 0\}$. Examples can be found in chapter 2 and in [11].

The first publication that we are aware of that deals with this class of systems is [6]. And the conjecture mentioned above can already be found in that paper. As the problem proved hard to solve, a number of investigations where carried out in the context of linear static inequalities, where the problem of minimal representations of systems containing both equalities and inequalities was solved completely [11]. As the study is placed in the context of behaviors, the Farkas Lemma for behavioral inequalities is also discussed in the Willem's Festschrift [12].

Until the Farkas Lemma for behavioral inequalities has been proven, issues like minimal representations, elimination of latent variables etcetera can not be solved in their full generality. It is our belief that the Farkas Lemma for behavioral inequalities as conjectured here, will be a cornerstone for further investigations in a theory for behavioral inequalities.

3.3 Available results

For the static case the conjecture is nothing else than the famous Farkas Lemma for linear inequalities. For the dynamic case, the conjecture holds true for a special case.

Proposition 3.2 *Let* $R \in \mathbb{R}^{g \times q}[s, s^{-1}]$ *be a full-row rank polynomial matrix. Let* $R' \in \mathbb{R}^{g' \times q}[s, s^{-1}]$. *Then:* $\{R(\sigma, \sigma^{-1})w \geq 0 \Rightarrow R'(\sigma, \sigma^{-1})w \geq 0\}$ *if and only if there exists a unique polynomial matrix* $H \in \mathbb{R}_+^{g' \times g}[s, s^{-1}]$ *such that* $R'(s, s^{-1}) = H(s, s^{-1})R(s, s^{-1})$. $\triangleleft$

The proof of this proposition can be found in [11, proposition 4.5.12].

3.4 A Related Conjecture

It is of interest to present a related conjecture, whose resolution is closely linked to the Farkas Lemma for behavioral inequalities.

Recall from [18] that a matrix $U \in \mathbb{R}^{g\times g}[s, s^{-1}]$ is said to be unimodular if it has an inverse $U^{-1} \in \mathbb{R}^{g\times g}[s, s^{-1}]$. We will call a matrix $H \in \mathbb{R}_+^{g\times g}[s, s^{-1}]$ posimodular if it is unimodular and $H^{-1} \in \mathbb{R}_+^{g\times g}[s, s^{-1}]$. Omitting the formal definitions, we will call a representation minimal if the number of equations used to describe the behavior is minimal.

Conjecture 3.3 *Let* $\{w \in (\mathbb{R}^q)^{\mathbb{Z}} | R_1(\sigma, \sigma^{-1})w = 0$ *and* $R_2(\sigma, \sigma^{-1})w \geq 0\}$ *and* $\{w \in (\mathbb{R}^q)^{\mathbb{Z}} | R_1'(\sigma, \sigma^{-1})w = 0$ *and* $R_2'(\sigma, \sigma^{-1})w \geq 0\}$ *both be two minimal representations. They represent the same behavior if and only if there are polynomial matrices* $U(s, s^{-1})$, $H(s, s^{-1})$ *and* $S(s, s^{-1})$ *such that*

$$\begin{bmatrix} R_1'(s, s^{-1}) \\ R_2'(s, s^{-1}) \end{bmatrix} = \begin{bmatrix} U(s, s^{-1}) & 0 \\ S(s, s^{-1}) & H(s, s^{-1}) \end{bmatrix} \begin{bmatrix} R_1(s, s^{-1}) \\ R_2(s, s^{-1}) \end{bmatrix} \tag{20}$$

with U *unimodular,* H *posimodular and no conditions on* S. $\triangleleft$

We remark that this conjecture holds true for static inequalities and for that case is given as proposition 3.4.5 in [11].

4 Concluding Remarks

In this report we have shown by illustrative examples that inequalities are often present in descriptions of systems with operational and environmental restrictions. The treatment of these kind of systems, both theoretically and by simulation, is hampered as a complete theory on system representations is lacking. One particular difficulty in establishing such a theory has been discussed. It has been conjectured that a Farkas Lemma holds for dynamical systems described by behavioral inequalities.

In the light of the many aerospace applications and the common mathematical denominator in these applications, theory building on efficient representations and simulation of constrained dynamical systems is an activity that warrants further attention in the Systems and Control Community.

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Appendix A: Notation

Below the notation that is used in this document is given. A short explanation is included. Equation refers to equalities and inequalities alike. All inequalities are componentwise.

$\mathbb{R}$	The reals.
$\mathbb{Z}$	The integers.
$\mathbb{Z}_+$	The nonnegative integers $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{R}_+$	The nonnegative reals $[0, \infty)$.
$\mathbb{R}^q$	The q-dimensional real vectors.
$\mathbb{R}^q_+$	The q-dimensional real vectors with nonnegative real coefficients.
$\mathbb{R}^{n\times m}$	The $(n \times m)$ matrices with real coefficients.
$\mathbb{R}^{n\times m}_+$	The $(n \times m)$ matrices with nonnegative real coefficients.
$\mathbb{R}[s, s^{-1}]$	The polynomials with real coefficients and positive and negative powers in the indeterminate s.
$\mathbb{R}^{n\times m}[s, s^{-1}]$	The $(n \times m)$ polynomial matrices with real coefficients and positive and negative powers in the indeterminate s.
$\mathbb{R}^{n\times m}_+[s, s^{-1}]$	Idem as $\mathbb{R}^{n\times m}[s, s^{-1}]$ but with nonnegative coefficients only.
I	The identity matrix.
A^T	The transpose of matrix A.
$A \geq 0$	$(A_{ij}) \geq 0$ for all i, j.
σ^t	The (backward) t-shift.
$R(\sigma, \sigma^{-1})$	A polynomial operator in the shift.
$\mathfrak{B}$	A behavior.