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Executive summary

Joint Integrated PDA Avoiding Track Coalescence under non-homogeneous clutter density

Problem area
In multi target tracking, measurements may originate from targets, whose existence and trajectory are generally not known a priori as well as from other random sources, usually termed clutter. Target measurements are only present in each scan with some probability of detection. In a multitarget situation, the measurements may also have originated from one of various targets. The number of targets in the surveillance area is unknown. Automatic tracking in this environment initiates and maintains tracks using both target and clutter measurements. If a track follows a target, it is called a true track otherwise it is called a false track. To discriminate between true and false tracks, an appropriate track quality measure has to be estimated simultaneously with track maintenance.

Description of work
Joint Probabilistic Data Association (JPDA) has proven to be effective in tracking multiple targets from measurements amidst clutter and missed detections. Joint Integrated PDA (JIPDA) has built upon this by including the probability of target existence as a track quality measure to enable automatic tracking and track maintenance. Both JPDA and JIPDA suffer from the problem of track coalescence of near target tracks. JPDA* is an extension of JPDA which avoids coalescence by pruning specific permutation hypotheses prior to hypothesis merging. Following JPDA*'s descriptor system derivation, this paper develops JIPDA*, an extension of JIPDA which avoids track coalescence. JIPDA* updates the probability of target existence as the track quality measure. An initial simulation study corroborates the effectiveness of this approach for tracking crossing targets in heavy clutter.

Results and conclusions
Through initial Monte Carlo simulations with IPDA, JIPDA and JIPDA* on an illustrative example, the coalescence avoidance property of JIPDA* has been confirmed, and the potential benefits of using JIPDA* in difficult target crossing scenarios shown.

Applicability
The applicability of the work comprises the implementation of the resulting filtering algorithms in a multitarget tracker, in particular the Advanced suRveillance Tracker And Server ARTAS.

Report no.
NLR-TP-2006-690

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Report classification
Unclassified

Date
September 2006

Knowledge area(s)
Advanced (sensor-) information processing

Descriptor(s)
Descriptor system
Bayesian estimation
False measurements
Missing measurements
Multitarget tracking
Probabilistic Data Association

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Joint Integrated PDA Avoiding Track Coalescence under Non-Homogeneous Clutter Density

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This report contains a paper presented at International Conference on Information Fusion, Florence, Italy, 10-13 July 2006.

Customer:	National Aerospace Laboratory NLR
Working Plan number:	2005 AT.1.C
Owner:	National Aerospace Laboratory NLR
Division:	Air Transport
Distribution:	Unlimited
Classification title:	Unclassified
	September 2006

Approved by:

Author	Reviewer	Managing department
8-12-06	Anonymous peer reviewers	22/12/06

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Abstract - ***Joint PDA has proven to be effective in tracking multiple targets from measurements amidst clutter and missed detections. Joint IPDA has built upon this by including the probability of target existence as a track quality measure to enable automatic tracking and track maintenance. Both JPDA and JIPDA suffer from the problem of track coalescence of near target tracks. JPDA* is an extension of JPDA which avoids coalescence by pruning specific permutation hypotheses prior to hypothesis merging. Following JPDA*'s descriptor system derivation, this paper developes JIPDA*, an extension of JIPDA which avoids track coalescence. JIPDA* updates the probability of target existence as the track quality measure. An initial simulation study corroborates the effectiveness of this approach for tracking crossing targets in heavy clutter.***

Keywords: Descriptor system, Bayesian estimation, False measurements, Missing measurements, Multitarget tracking, Probabilistic Data Association

1 Introduction

In multi target tracking, measurements (detections) may originate from targets, whose existence and trajectory are generally not known *a priori* as well as from other random sources, usually termed clutter. Target measurements are only present in each scan with some probability of detection. In a multi-target situation, the measurements may also have originated from one of various targets. The number of targets in the surveillance area is unknown. Automatic tracking in this environment initiates and maintains tracks using both target and clutter measurements. If a track follows a target, we call it a true track otherwise we call it a false track. To discriminate between true and false tracks, an appropriate track quality measure has to be estimated simultaneously with track maintenance. Moreover the possibility that a measurement may have originated from another target not being followed by the current track has to be taken into account.

Single target tracking algorithms which use an appropriate track quality measure are for example IPDA and related algorithms [1]-[4], GPB1-PDA [5], [6] and IMM-PDA [7]. Multi-target tracking algorithms allow for the possibility that measurements may have arisen from the targets being followed by other tracks. An optimum approach forms all possible joint measurement-to-track assignment hypotheses and recursively calculates their *a posteriori* probabilities. Joint PDA (JPDA) [8], [9] is one of the best known algorithms following this approach. JPDA incorporates all-neighbours single Gaussian probability density function PDA approximation with the Bayesian data association paradigm. However, JPDA essentially assumes that all tracks are true tracks. Using the probability of target existence paradigm [1], JIPDA [10] integrates the probability of target existence estimation with JPDA track updating.

JPDA and JIPDA show remarkable resistance to clutter and missed detections. However, they both tend to coalesce near tracks. A recent algorithm for multi target tracking in clutter, JPDA* [11] improves JPDA considerably in this respect, while still retaining resistance to clutter and missed detections. The aim of this paper is to combine the JIPDA development of [10] with the JPDA* development of [11]. In order to accomplish this, the JIPDA problem formulation is first integrated with the descriptor system approach of [11]. Subsequently the paper develops JIPDA* algorithm for tracking multiple targets in non-homogeneous clutter, with integrated track quality measure and with track coalescence resistance of JPDA*. Similar as in [12], we drop the homogeneous false plot density assumption[1]. The motivation for doing so stems from the convincing demonstration in [13] that there is sound reason for modern signal processing and tracking designs to say goodbye to the classical Newman-Pearson criterion of fixed probability of false alerts.

The paper is organized as follows. Section 2 defines the problem considered. Section 3 embeds the tracking problem into one of filtering for a jump linear descriptor system with stochastic i.i.d. (independent identically distributed) coefficients. Section 4 develops exact Bayesian and JPDA* filter equations. Section 5 illustrates the advantages of JIPDA* using an initial simulation study. Section 6 draws conclusions.

[1] In [19] we started the JIPDA* development for the simpler situation of homogeneous clutter density.

2 Stochastic Models

This section describes the stochastic models used for target existence, potential targets and measurements.

2.1 Target existence

Tracks may be established using target measurements, or they can start following clutter measurements. Thus, the existence of a target being followed by each track is a random event. We note two models for target existence propagation [1]. Markov Chain One model assumes that existing target is always detectable. Markov Chain Two model also allows for the possibility that the target exists and is temporarily not detectable. In this text we will derive formulae for the case of Markov Chain One model for track existence propagation. For each track i, random event $\xi_{i,t}$ will describe target existence at time t:

$$\begin{aligned} \xi_t^i &= 1 \quad \text{if the target exists} \\ \xi_t^i &= 0 \quad \text{if the target does not exist.} \end{aligned}$$

Markov Chain One transition probability of target existence for track i satisfies

$$\begin{aligned} &P\{\xi_t^i = 1|\xi_{t-1}^i = 1\} = p_{11} \in (0,1) \\ &P\{\xi_t^i = 0|\xi_{t-1}^i = 1\} = 1 - p_{11} \\ &P\{\xi_t^i = 1|\xi_{t-1}^i = 0\} = 0 \\ &P\{\xi_t^i = 0|\xi_{t-1}^i = 0\} = 1 \end{aligned}$$

The last two equations mean that from the moment t on that an existing target i becomes non-existing, it will remain forever non-existing, i.e. if $\xi_t^i = 0$ then $\xi_s^i = 0$ for all $s \geq t$.

2.2 Potential targets

Consider $M_t = M$ potential targets (tracks) at moment t. Denote with $\xi_t \triangleq \text{Col}\{\xi_{1,t}, \cdots, \xi_{M,t}\}$ target existence indicator vector at moment t. Assume that the state of the i-th potential target satisfies:

$$x_t^i = a^i x_{t-1}^i + b^i w_t^i, \quad i = 1, \cdots, M, \tag{1}$$

where x_t^i is the n-vectorial state of the i-th potential target, a^i and b^i are $(n \times n)$- and $(n \times n')$-matrices, and w_t^i is a sequence of i.i.d. standard Gaussian variables of dimension n' with w_t^i , w_t^j independent for all $i \neq j$ and w_t^i ,x_0^i, x_0^j independent for all $i \neq j$. Let $x_t = \text{Col}\{x_t^1, \cdots, x_t^M\}$, $A = \text{Diag}\{a^1, \cdots, a^M\}$, $B = \text{Diag}\{b^1, \cdots, b^M\}$, and $w_t = \text{Col}\{w_t^1, \cdots, w_t^M\}$. If we assume that from moment t to moment $t+1$, none of the tracks is deleted and there are no track births, then $M_{t+1} = M_t = M$ and the state of our M potential targets evolves as follows:

$$x_t = Ax_{t-1} + Bw_t \tag{2}$$

with A of size $Mn \times Mn$ and B of size $Mn \times Mn'$.

2.3 Measurements

A set of measurements consists of potential target measurements and clutter measurements.

2.3.1 Potential target measurements

We assume that associated with state x_t^i is a potential measurement z_t^i satisfying:

$$z_t^i = h^i x_t^i + g^i v_t^i, \quad i = 1, \cdots, M \tag{3}$$

where z_t^i is an m-vector, h^i is an $(m \times n)$-matrix and g^i is an $(m \times m')$-matrix, and v_t^i is a sequence of i.i.d. standard Gaussian variables of dimension m' with v_t^i and v_t^j independent for all $i \neq j$. Moreover v_t^i is independent of x_0^j and w_t^j for all i,j. Next with $z_t = \text{Col}\{z_t^1, \cdots, z_t^M\}$, $H = \text{Diag}\{h^1, \cdots, h^M\}$, $G = \text{Diag}\{g^1, \cdots, g^M\}$, and $v_t = \text{Col}\{v_t^1, \cdots, v_t^M\}$:

$$z_t = Hx_t + Gv_t \tag{4}$$

with H and G of size $Mm \times Mn$ and $Mm \times Mm'$ respectively. Because of notational simplicity we assume h^i, g^i, H and G to be time-invariant.

2.3.2 Existing target detections

We next introduce a model that takes into account that not all targets have to be detected at moment t, which implies that not all potential measurements z_t^i have to be available as true measurements at moment t. To this end, let $\phi_{i,t} \in\{0,1\}$ be the existence and detection indicator for potential target i, which satisfies:

$$\begin{aligned} \phi_{i,t} &= 1 \quad \text{if } \xi_{1,t} = 1 \ \underline{\text{and}}\ z_t^i \text{ is detected} \\ \phi_{i,t} &= 0 \quad \text{if } \xi_{1,t} = 1 \ \underline{\text{and}}\ z_t^i \text{ is undetected} \\ \phi_{i,t} &= 0 \quad \text{if } \xi_{1,t} = 0 \end{aligned}$$

We capture this through the following equation

$$\phi_{i,t} = \xi_{i,t}\delta_{i,t} \tag{5.a}$$

with the $(0,1)$ valued $\{\delta_{i,t}\}$ a sequence of i.i.d. random variables satisfying

$$\begin{aligned} \text{Prob}\{\delta_{i,t} = 1\} &= P_{\text{d}}^i \in (0,1) \\ \text{Prob}\{\delta_{i,t} = 0\} &= 1 - P_{\text{d}}^i \end{aligned}$$

where P_{d}^i is the conditional detection probability of potential target i given target i exists. Hence the conditional probability distribution of ϕ_t^i given $\xi_t^i = \xi^i \in \{0,1\}$ satisfies

$$P\{\phi_t^i = \phi^i|\xi_t^i = \xi^i\} = (1 - \xi^i P_{\text{d}}^i)^{1-\phi^i}(\xi^i P_{\text{d}}^i)^{\phi^i} \tag{5.b}$$

This approach yields the following existence and detection indicator vector ϕ_t of size M:

$$\phi_t = \text{Col}\{\phi_{1,t}, ..., \phi_{M,t}\}.$$

The number of existing and detected targets is

$$D_t \triangleq \sum_{i=1}^{M} \phi_{i,t} = \xi_t^T \delta_t$$

In order to link the existence and detection indicator vector with the measurement model, we introduce the following operator Φ: for an arbitrary (0,1)-valued M'-vector ϕ' we define $D(\phi') \triangleq \sum_{i=1}^{M'} \phi'_i$ and the operator Φ producing $\Phi(\phi')$ as a $(0,1)$-valued matrix of size $D(\phi') \times M'$ of which the ith row equals the ith non-zero row of Diag$\{\phi'\}$. Next we define, for $D_t > 0$, a vector that contains all measurements originating from targets at moment t in a fixed order.

$$\tilde{z}_t \triangleq \underline{\Phi}(\phi_t) z_t, \text{ where } \underline{\Phi}(\phi_t) \triangleq \Phi(\phi_t) \otimes I_m,$$

with I_m a unit-matrix of size m, and $\otimes$ the Kronecker product. In reality we do not know the order of the targets. Hence, we introduce the stochastic $D_t \times D_t$ permutation matrix χ_t, which is conditionally independent of $\{\phi_t\}$. We also assume that $\{\chi_t\}$ is a sequence of independent matrices. Hence, for $D_t > 0$,

$$\tilde{\tilde{z}}_t \triangleq \underline{\chi}_t \tilde{z}_t, \text{ where } \underline{\chi}_t \triangleq \chi_t \otimes I_m,$$

is a vector that contains all measurements originating from targets at moment t in a random order.

2.3.3 Measurements originating from clutter

Denote by Poisson random variable F_t the number of false measurements at moment t:

$$\begin{aligned} p_{F_t}(F) &= \tfrac{(\hat{F}_t)^F}{F!} \exp\big(-\hat{F}_t\big), \qquad F \geq 0 \\ &= 0, \qquad\qquad\qquad\qquad\quad \text{else} \end{aligned} \tag{6.a}$$

where $\hat{F}_t$ is the expected number of false measurements. A column-vector f_t given F_t i.i.d. false measurements has the following pdf:

$$p_{f_t|F_t}(f|F) = \prod_{i=1}^{F} p_f(f^i). \tag{6.b}$$

where $p_f(\cdot)$ is the pdf of a false measurement. Hence the local density $\lambda(\cdot)$ of false measurements satisfies:

$$\lambda(f^i) = \hat{F}_t p_f(f^i) \tag{6.c}$$

Furthermore we assume that the process $\{F_t, f_t\}$ is a sequence of independent vectors, which are independent of $\{x_t\}, \{w_t\}, \{v_t\}$ and $\{\phi_t\}$.

2.3.4 Insertion of clutter measurements

Let the random variable L_t be the total number of measurements at moment t. Thus,

$$L_t = D_t + F_t$$

With $\tilde{\mathrm{y}}_t \triangleq \mathrm{Col}\{\tilde{\tilde{z}}_t, v_t^*\}$, it follows with the above defined variables that

$$\tilde{\mathrm{y}}_t = \begin{bmatrix} \underline{\chi}_t \underline{\Phi}(\phi_t) z_t \\ \cdots\cdots\cdots \\ v_t^* \end{bmatrix}, \text{ if } L_t > D_t > 0 \tag{7}$$

where the upper and lower subvector parts disappear for $D_t = 0$ and $L_t = D_t$ respectively. The clutter measurements remain to be randomly inserted between the measurements originating from the detected targets. Define target and clutter indicator processes, denoted by $\{\psi_t\}$ and $\{\psi_t^*\}$, respectively:

$$\begin{aligned} \psi_t^i &= 1 \quad \text{if measurement } i \text{ is detection} \\ \psi_t^i &= 0 \quad \text{if measurement } i \text{ is clutter} \\ \psi_t^{*i} &= 1 \quad \text{if measurement } i \text{ is clutter} \\ \psi_i^{*i} &= 0 \quad \text{if measurement } i \text{ is detection} \end{aligned}$$

Thus $\psi_{i,t}^* = 1 - \psi_{i,t}$, and

$$\begin{aligned} \psi_t &\triangleq \mathrm{Col}\{\psi_{1,t}, \cdots, \psi_{L_t,t}\} \\ \psi_t^* &\triangleq \mathrm{Col}\{\psi_{1,t}^*, \cdots, \psi_{L_t,t}^*\}. \end{aligned}$$

The measurement vector with clutter inserted is:

$$\mathrm{y}_t = \left[\underline{\Phi}(\psi_t)^T \vdots \underline{\Phi}(\psi_t^*)^T \right] \tilde{\mathrm{y}}_t \text{ if } L_t > D_t > 0 \tag{8}$$

Substituting (7) into (8) yields the following model for the observation vector y_t if $L_t > D_t > 0$:

$$\mathrm{y}_t = \left[\underline{\Phi}(\psi_t)^T \vdots \underline{\Phi}(\psi_t^*)^T \right] \begin{bmatrix} \underline{\chi}_t \underline{\Phi}(\phi_t) z_t \\ \cdots\cdots\cdots \\ v_t^* \end{bmatrix} \tag{9}$$

This, together with equations (2) and (4), forms a complete characterization of the multitarget scenario in terms of a system of stochastic difference equations.

3 Descriptor system embedding

Because $\left[\underline{\Phi}(\psi_t)^T \vdots \underline{\Phi}(\psi_t^*)^T \right]$ is a permutation matrix for $L_t > D_t > 0$, its inverse equals its transpose:

$$\left[\underline{\Phi}(\psi_t)^T \vdots \underline{\Phi}(\psi_t^*)^T \right]^{-1} = \begin{bmatrix} \underline{\Phi}(\psi_t) \\ \cdots \\ \underline{\Phi}(\psi_t^*) \end{bmatrix} \tag{10}$$

Premultiplying (9) by such inverse yields

$$\begin{bmatrix} \underline{\Phi}(\psi_t) \\ \cdots \\ \underline{\Phi}(\psi_t^*) \end{bmatrix} \mathrm{y}_t = \begin{bmatrix} \underline{\chi}_t \underline{\Phi}(\phi_t) z_t \\ \cdots\cdots\cdots \\ v_t^* \end{bmatrix} \quad \text{if } L_t > D_t > 0 \tag{11}$$

From (11), it follows that

$$\underline{\Phi}(\psi_t) \mathrm{y}_t = \underline{\chi}_t \underline{\Phi}(\phi_t) z_t \text{ if } D_t > 0 \tag{12}$$

Substitution of (4) into (12) yields:

$$\underline{\Phi}(\psi_t) \mathrm{y}_t = \underline{\chi}_t \underline{\Phi}(\phi_t) H x_t + \underline{\chi}_t \underline{\Phi}(\phi_t) G v_t \text{ if } D_t > 0 \tag{13}$$

Notice that (13) is a linear Gaussian descriptor system [14] with stochastic i.i.d. coefficients $\underline{\Phi}(\psi_t)$ and $\underline{\chi}_t \underline{\Phi}(\phi_t)$. Because χ_t has an inverse, (13) becomes

$$\underline{\chi}_t^T \underline{\Phi}(\psi_t) \mathrm{y}_t = \underline{\Phi}(\phi_t) H x_t + \underline{\Phi}(\phi_t) G v_t \text{ if } D_t > 0 \tag{14}$$

Next we introduce an auxiliary indicator matrix process $\tilde{\chi}_t$ of size $D_t \times L_t$, as follows:

$$\tilde{\chi}_t \triangleq \chi_t^T \Phi(\psi_t) \text{ if } D_t > 0.$$

With this we get a simplified version of (14):

$$\underline{\tilde{\chi}}_t \mathrm{y}_t = \underline{\Phi}(\phi_t) H x_t + \underline{\Phi}(\phi_t) G v_t \text{ if } D_t > 0 \tag{15}$$

where $\underline{\tilde{\chi}} \triangleq \tilde{\chi} \otimes I_m$. Size of $\underline{\tilde{\chi}}_t$ is $D_t m \times L_t m$ and size of $\underline{\Phi}(\phi_t)$ is $D_t m \times M m$.

4 Exact and JIPDA* filter equations

In this section we present a Bayesian characterization of the track state in (2), conditional on the σ-algebra generated by measurements y_t up to and including moment t, denoted here by Y_t. From (15), it follows that for $D_t > 0$ all relevant associations and permutations can be covered by $(\phi_t, \tilde{\chi}_t)$-hypotheses. We extend this to $D_t = 0$ by adding the combination $\phi_t = \{0\}^M$ and $\tilde{\chi}_t = \{\}^{L_t}$. Through defining the weights

$$\beta_t(\xi,\phi,\tilde{\chi}) \triangleq P\{\xi_t = \xi, \phi_t = \phi, \tilde{\chi}_t = \tilde{\chi}|Y_t\},$$

the law of total probability yields:

$$P\{\xi_t^i = 1|Y_t\} = \sum_{\substack{\xi,\phi,\tilde{\chi}\\ \xi^i=1}} \beta_t(\xi,\phi,\tilde{\chi}) \qquad (16)$$

$$p_{x_t^i|\xi_t^i=1,Y_t}(x^i) = \sum_{\tilde{\chi},\phi} P\{\phi_t = \phi, \tilde{\chi}_t = \tilde{\chi}|\xi_t^i = 1, Y_t\} \cdot \cdot p_{x_t^i|\xi_t^i,\phi_t,\tilde{\chi}_t,Y_t}(x^i|1,\phi,\tilde{\chi}) \qquad (17)$$

We characterize the terms in the last summation.

$$P\{\phi_t = \phi, \tilde{\chi}_t = \tilde{\chi}|\xi_t^i = 1, Y_t\} = \frac{P\{\phi_t = \phi, \tilde{\chi}_t = \tilde{\chi}, \xi_t^i = 1|Y_t\}}{P\{\xi_t^i = 1|Y_t\}}$$

$$P\{\phi_t = \phi, \tilde{\chi}_t = \tilde{\chi}, \xi_t^i = 1|Y_t\} = \sum_{\xi:\xi^i=1} \beta_t(\xi,\phi,\tilde{\chi})$$

Proposition 1 *For any $\xi,\phi \in\{0,1\}^M$, such that $D(\phi) \le L_t$, and any $\tilde{\chi}_t$ matrix realization $\tilde{\chi}$ of size $D(\phi)\times L_t$, the following holds true:*

$$p_{x_t|\xi_t,\phi_t,\tilde{\chi}_t,Y_t}(x \mid \xi,\phi,\tilde{\chi}) = = \frac{p_{\tilde{z}_t|x_t,\xi_t,\phi_t}(\underline{\tilde{\chi}}\mathrm{y}_t \mid x,\xi,\phi)\cdot p_{x_t|\xi_t,Y_{t-1}}(x \mid \xi)}{F_t(\xi,\phi,\tilde{\chi})} \qquad (18)$$

$$\beta_t(\xi,\phi,\tilde{\chi}) = F_t(\xi,\phi,\tilde{\chi}) \prod_{j=1}^{L_t-D(\phi)} \lambda\left(\left[\Phi\left(1_{L_t} - \tilde{\chi}^T\tilde{\chi}1_{L_t}\right)\mathrm{y}_t\right]_j\right)\cdot \cdot[\prod_{i=1}^{M}(1-\xi^i P_{\mathrm{d}}^i)^{(1-\phi^i)}(\xi^i P_{\mathrm{d}}^i)^{\phi^i}]\cdot p_{\xi_t|Y_{t-1}}(\xi)/c_t \qquad (19)$$

where $\underline{\tilde{\chi}} \triangleq \tilde{\chi}\otimes I_m$, $1_{L_t} = [1,...,1]^T$ is an L_t vector with 1-valued elements and $F_t(\xi,\phi,\tilde{\chi})$ and c_t normalize $p_{x_t|\xi_t,\phi_t,\tilde{\chi}_t,Y_t}(x|\xi,\phi,\tilde{\chi})$ and $\beta_t(\xi,\phi,\tilde{\chi})$ respectively.

Proof: See Appendix A.

Next we assume per track independent target existence and state density given existence:

$$p_{\xi_t|Y_{t-1}}(\xi) = \prod_{i=1}^{M} p_{\xi_t^i|Y_{t-1}}(\xi^i)$$

$$p_{x_t|\xi_t,Y_{t-1}}(x|\xi) = \prod_{i=1}^{M} p_{x_t^i|\xi_t^i,Y_{t-1}}(x^i|\xi^i)$$

This leads to the following Theorem and Corollary.

Theorem 1 *Let $p_{\xi_t|Y_{t-1}}(\xi) = \prod_{i=1}^M p_{\xi_t^i|Y_{t-1}}(\xi^i)$ and let $p_{x_t|\xi_t,Y_{t-1}}(x|\xi) = \prod_{i=1}^M p_{x_t^i|\xi_t^i,Y_{t-1}}(x^i|\xi^i)$, then $\beta_t(\xi,\phi,\tilde{\chi})$ of proposition 1 satisfies:*

$$\beta_t(\xi,\phi,\tilde{\chi}) = \prod_{j=1}^{L_t-D(\phi)} \lambda\left(\left[\Phi\left(1_{L_t} - \tilde{\chi}^T\tilde{\chi}1_{L_t}\right)\mathrm{y}_t\right]_j\right)\cdot \cdot\prod_{i=1}^{M}\left[f_t^i(\phi,\tilde{\chi})(1-\xi^i P_{\mathrm{d}}^i)^{(1-\phi^i)}(\xi^i P_{\mathrm{d}}^i)^{(\phi^i)}\cdot p_{\xi_t^i|Y_{t-1}}(\xi^i)\right]/c_t$$

with for $\phi^i = 0$: $f_t^i(\phi,\tilde{\chi}) = 1$, and for $\phi^i = 1$

$$f_t^i(\phi,\tilde{\chi}) = \int_{\mathbb{R}^n} p_{z_t^i|x_t^i,\phi_t}\left(\left[\Phi(\phi)^T\tilde{\chi}\right]_{ik}\mathrm{y}_t^k|x^i,\phi\right)\cdot \cdot p_{x_t^i|\xi_t^i,Y_{t-1}}(x^i|1)dx^i$$

Moreover

$$p_{x_t|\xi_t,\phi_t,\tilde{\chi}_t,Y_t}(x|\xi,\phi,\tilde{\chi}) = \prod_{i=1}^{M} p_{x_t^i|\xi_t^i,\phi_t,\tilde{\chi}_t,Y_t}(x^i|\xi^i,\phi,\tilde{\chi})$$

with:

$$p_{x_t^i|\xi_t^i,\phi_t,\tilde{\chi}_t,Y_t}(x^i|\xi^i,\phi,\tilde{\chi}) = = \frac{p_{z_t^i|x_t^i,\phi_t}\left(\left[\Phi(\phi)^T\tilde{\chi}\right]_{ik}\mathrm{y}_t^k|x^i,\phi\right)\cdot p_{x_t^i|\xi_t^i,Y_{t-1}}(x^i|1)}{f_t^i(\phi,\tilde{\chi})} \quad \text{if } \phi^i = 1 \text{ and } \xi^i = 1$$

$$= p_{x_t^i|\xi_t^i,Y_{t-1}}(x^i|\xi^i) \quad \text{if } \phi^i = 0 \text{ and/or } \xi^i = 0$$

Proof: Omitted

Corollary 1 *Under the assumptions of Theorem 1, for each potential target i, the pdf of trajectory state estimate, $p_{x_t^i|\xi_t^i=1,Y_t}$ is a mixture of pdf's of trajectory state estimates, each calculated assuming one of the measurements is the detection of potential target i:*

$$p_{x_t^i|\xi_t^i,Y_t}(x^i|1) = \sum_{k=1}^{L_t} p_{x_t^i|\xi_t^i,z_t^i,Y_{t-1}}(x^i|1,\mathrm{y}_t^k)\cdot\beta_t^{ik} + +p_{x_t^i|\xi_t^i,Y_{t-1}}(x^i|1)\cdot\beta_t^{i0} \qquad (20)$$

where $p_{x_t^i|\xi_t^i,z_t^i,Y_{t-1}}(x^i|1,\mathrm{y}_t^k)$ is the estimation pdf of potential target i given that it exists and that its' detection at time t was y_t^k. β_t^{ik} is the a posteriori probability that measurement k is a detection of an existing target i; and β_t^{i0} is the a posteriori probability that there is no detection of potential target i, given target i existence:

$$\beta_t^{ik} \triangleq P\{[\Phi(\phi_t)^T\tilde{\chi}_t]_{ik} = 1|\xi_t^i = 1, Y_t\} = \qquad (21)$$
$$= \sum_{\substack{\phi,\tilde{\chi}\\ \xi:\xi^i=1}} [\Phi(\phi)^T\tilde{\chi}]_{ik}\beta_t(\xi,\phi,\tilde{\chi})/P\{\xi_i = 1|Y_t\}$$

$$\beta_t^{i0} \triangleq P\{\phi_t^i = 0|\xi_t^i = 1, Y_t\} = \qquad (22)$$
$$= \sum_{\substack{\tilde{\chi}\\ \phi:\phi^i=0\\ \xi:\xi^i=1}} \beta_t(\xi,\phi,\tilde{\chi})/P\{\xi_i = 1|Y_t\}$$

where $\beta_t(\xi,\phi,\tilde{\chi})$ is given in Theorem 1 for non-homogeneous λ, and $[\Phi(\phi_t)^T\tilde{\chi}_t]_{ik} = 1$ under the hypotheses that potential target i exists, is detected, and measurement k applies.

Proof: Omitted

4.1 JIPDA (Non-homogeneous λ)

JIPDA approximates each track trajectory estimate pdf with a Gaussian function [12], i.e.:

$$\begin{aligned} p_{x_t^i|\xi_t^i,Y_{t-1}}(x^i|1) &\approx N(x^i;\bar{x}_t^i,\bar{P}_t^i) \\ p_{x_t^i|\xi_t^i,Y_t}(x^i|1) &\approx N(x^i;\hat{x}_t^i,\hat{P}_t^i). \end{aligned}$$

Thus, $p_{x_t|\xi_t^i,Y_{t-1}}(x|1)$ is Gaussian with mean $\bar{x}_t = \mathrm{Col}\{\bar{x}_t^1,\cdots,\bar{x}_t^M\}$ and covariance $\bar{P}_t = \mathrm{Diag}\{\bar{P}_t^1,\cdots,\bar{P}_t^M\}$. Then $p_{x_t^i|\xi_{i,t}^i,Y_t}(x^i|1)$ is a Gaussian mixture, approximated with a single Gaussian which will preserve the overall mean $\hat{x}_t^i$ and its overall covariance $\hat{P}_t^i$:

$$\hat{x}_t^i = \bar{x}_t^i + W_t^i \cdot \left(\sum_{k=1}^{L_t} \beta_t^{ik}\mu_t^{ik}\right) \quad (23)$$

$$\begin{aligned} \hat{P}_t^i = {} & \bar{P}_t^i - W_t^i h^i \bar{P}_t^i \left(\sum_{k=1}^{L_t} \beta_t^{ik}\right) + \quad (24) \\ & W_t^i \left(\sum_{k=1}^{L_t} \beta_t^{ik}\mu_t^{ik}(\mu_t^{ik})^T\right) \cdot (W_t^i)^T - \\ & W_t^i \left(\sum_{k=1}^{L_t} \beta_t^{ik}\mu_t^{ik}\right) \cdot \left(\sum_{k'=1}^{L_t} \beta_t^{ik'}\mu_t^{ik'}\right)^T (W_t^i)^T \end{aligned}$$

with β_t^{ik} as given in Corollary 1 and Theorem 1 (for non-homogeneous λ), and

$$\begin{aligned} W_t^i &= \bar{P}_t^i(h^i)^T[h^i\bar{P}_t^i(h^i)^T + g^i(g^i)^T]^{-1} \\ \mu_t^{ik} &= y_t^k - h^i\bar{x}_t^i \end{aligned}$$

4.2 JIPDA* (Non-homogeneous λ)

A shortcoming of JPDA and JIPDA is its sensitivity to track coalescence. Following the approach of JPDA* [11], JIPDA* filter equations are obtained from JIPDA algorithm by keeping only the strongest hypotheses with the common (ϕ,ψ), prior to updating track state, probability of target existence and estimate of trajectory state conditioned on target existence. In other words, keep the strongest hypotheses from each set of hypotheses having common set of detected tracks and allocated measurements. For every ϕ and ψ find

$$\hat{\chi}_t(\xi,\phi,\psi) \stackrel{\triangle}{=} \underset{\chi}{\mathrm{Argmax}}\ \beta_t(\xi,\phi,\chi^T\Phi(\psi))$$

where the maximization is over all permutation matrices χ of size $D(\phi)\times D(\phi)$, and $\beta_t(\xi,\phi,\tilde{\chi})$ given in Theorem 1 for non-homogeneous λ. Then the following values for data association probabilities are used

$$\hat{\beta}_t(\xi,\phi,\chi^T\Phi(\psi)) = \begin{cases} \hat{c}_t^{\,-1}\beta_t(\xi,\phi,\chi^T\Phi(\psi)), & \chi_t = \hat{\chi}_t(\phi,\psi) \\ 0, & \text{otherwise} \end{cases}$$

with $\hat{c}_t$ a normalization constant such that

$$\sum_{\xi,\phi,\psi} \hat{\beta}_t(\xi,\phi,\chi^T\Phi(\psi)) = 1$$

Subsequently the β_t^{ik}'s are evaluated using these $\hat{\beta}_t(\xi,\phi,\tilde{\chi})$'s.

5 Initial Simulation Study

The purpose of these simulations is to initially compare the JIPDA* algorithm with JIPDA [10] and IPDA [1]. The focus of this study is on the false track discrimination and target crossing outcomes, in a heavy and non-homogeneous clutter environment. Non parametric versions [10] of the algorithms are used.

A two-dimensional surveillance situation was considered. The area under surveillance was $1000m$ long and $400m$ wide. The false measurements satisfied a Poisson distribution with density $1.0\cdot 10^{-4}$ /scan /m^2.

The experiments consisted of 1000 runs, with each run consisting of 50 scans. There are two targets in the surveillance region whose trajectories cross at scan 35 with a crossing angle of 10^o. Targets appear in scan one and move with uniform motion, with target one having an initial state of $x_0 = [130m\ 15m/s\ 200m\ 0m/s]$. The other target also moves with uniform motion at a speed of $15m/s$.

The motion of each target is modeled in Cartesian coordinates as

$$x_t^i = a x_{t-1}^i + w_t^i \quad (25)$$

where x_t^i is the target state vector at time t and consists of the position and the velocity in each of the 2 coordinates

$$x' = \left[\begin{array}{cccc} x & \dot{x} & y & \dot{y} \end{array}\right] \quad (26)$$

with the transition matrix a

$$a = \left[\begin{array}{cc} a_T & 0 \\ 0 & a_T \end{array}\right]; \qquad a_T = \left[\begin{array}{cc} 1 & T \\ 0 & 1 \end{array}\right] \quad (27)$$

where T is the sampling period of 1s. The plant noise w_t is the zero mean white Gaussian noise with known covariance

$$E\left[w_t^i w_s^{iT}\right] = Q\,\delta\,(t-s) \quad (28)$$

where δ is the Dirac function and

$$Q = q\left[\begin{array}{cc} Q_T & 0 \\ 0 & Q_T \end{array}\right];\ Q_T = \left[\begin{array}{cc} T^4/4 & T^3/2 \\ T^3/2 & T^2 \end{array}\right] \quad (29)$$

with $q = 0.75$. The detection probability was 0.9 throughout the experiment and the sensor introduced independent errors in the x and y coordinates with a root mean square of $5m$ in each coordinate. The tracking estimation filter was a simple Kalman filter based on the described trajectory and sensor models. The selection probability was set to $P_W = 0.99$.

The JIPDA / JIPDA* derivation does not address the coordination between track formation (initiation), confirmation, maintenance, termination and merging. In these experiments, tracks are initiated automatically in every scan, using two point differencing and initial track probability assignment as described in [4]. A track is initiated by any two measurements in two consecutive scans, unless a maximum speed test is not satisfied. Thus a significant number of false tracks are initialized in every scan of every run. Two point differencing is employed to determine the state of the new track after the second scan. Initial probability of existence is assigned to the first measurement of the pair, after which it propagates according to the IPDA formulae. This probability is the same for each first measurement, however in heavy clutter this is usually distributed to many second scan measurements. Thus in heavy clutter each track starts life with disadvantaged probability of target existence.

Each new track is deemed false with respect to all existing targets. A false track becomes a true track with respect to a target if its state estimation error (2D in both position and speed) becomes small enough. Once a track is a true track with respect to a target, it remains true for as long as it keeps on selecting the target's detections when they exist. Thus a true track may have a considerable error.

A track propagates according to IPDA until confirmation, after which IPDA, JIDPA or JIDPA* formulae are applied. A track gets confirmed once his probability of target existence reaches the confirmation threshold (same value for each algorithm), and stays confirmed until termination. A track is terminated once his probability of target existence falls below a termination threshold, or it goes out of the surveillance area, or is merged with another track, or fails the maximum speed test. Two tracks are merged when their estimated position and speed difference falls below a certain threshold. Merging actually involves simply dropping the track with larger covariance matrix determinant. The confirmation status is obtained by OR operation of the two tracks confirmation statuses, as well as true track statuses with respect to all targets.

False tracks are carried over from one simulation run to the other, while the true tracks are terminated at the end of each simulation run. The sum of confirmed false track scans was approximately equal for each simulation experiment and in the vicinity of 1 per 800 scans in each of the simulation experiments.

The simulation results are presented in Table 1. The true track situation is observed on scan 20 and then again on scan 50 to evaluate the crossing results. Only cases where two confirmed tracks were following each of the two targets at scan 20 were considered. A confirmed track is counted false only if it is false with respect to all existing targets. In this experiment, JIPDA* was unambiguously better than JIPDA. The percentage of successful crossings was substantially higher. The frequency of track merging was order of magnitude smaller, indicating coalescence avoidance. Finally, IPDA did not have a single successful crossing case, showing the weakness of using single target tracking filter in complex multi target situations.

Table 1: Target crossing outcomes

	JIPDA*	JIPDA	IPDA
Total	816	815	811
Both OK	579	372	0
One OK	105	244	453
Both switch	76	2	0
One switch	51	196	358
Both lost	5	1	0
Merged	35	406	801

The true track confirmations are presented in Figure 1. Each curve shows the number of cases in which a confirmed track was following a target. For two targets and a thousand runs, 2000 indicates 100% success rate. The horizontal axis depicts the time in scans from the start of the simulation run. Again, JIPDA* shows the best performance, having lost only 5% of tracks after crossing, compared to 19% and 32% in the case of JIPDA of IPDA respectively. Figure 2 shows estimation errors over time. Again, JIPDA* shows the smallest estimation errors after the crossover.

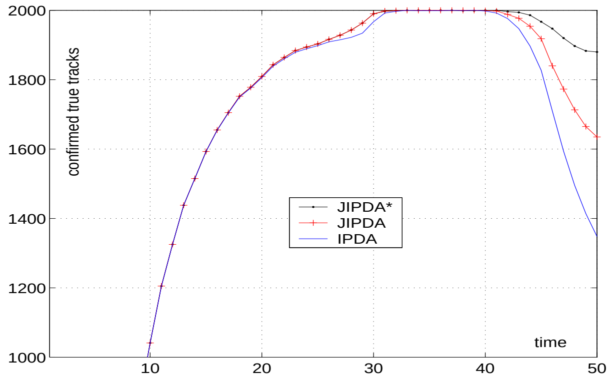

Figure 1: Confirmed target tracks

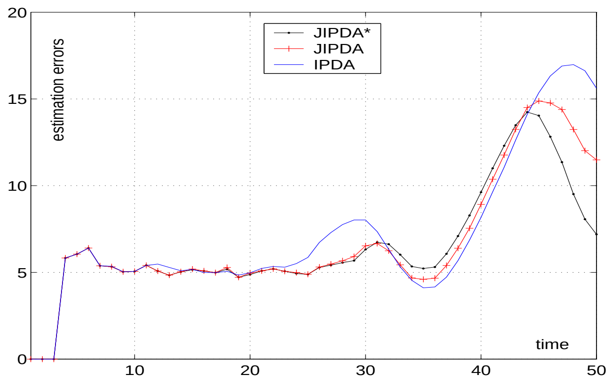

Figure 2: RMS errors

Though it is not directly visible from figures 1 and 2, the initial design, used to co-ordinate track initiation, merging etc., has some shortcomings. In particular a single measurement can be used to update many tracks independently in IPDA, or jointly in JIPDA or JIPDA*. In the JIPDA and JIPDA* simulations, unconfirmed tracks are processed by IDPA, and confirmed tracks are processed using JIPDA / JIPDA*. Thus, if a measurement is selected by 2 unconfirmed tracks and 3 confirmed tracks, it will be used for IPDA update of 2 unconfirmed tracks, and then for JIPDA or JIDPA* update of the 3 confirmed tracks separately. It may also be used to update a 2nd scan track initiation centered around a previous measurement. In addition a single measurement may be used for many 2nd scan initiations. Finally, the measurement is used as the first measurement for future track initiations. This explains the high numbers of merging events counted in the last row of Table 1. There is a need to significantly improve the coordination of track maintenance with track initiation, track merging and track termination. Subsequently a larger variety of target crossing scenarios should be considered.

6 Conclusions

This paper has formulated the multi target tracking including track existence estimation under nonhomogeneous false measurements within the descriptor system modelling approach of [11]. Subsequently this formulation has been used to characterize Bayesian filter recursions of the joint conditional density for multi target state and existence (Proposition 1 and Theorem 1). It has also been shown how the JIPDA filter equations of [10] can be obtained from this. Subsequently we applied the JPDA* hypotheses reduction approach of [11] to the descriptor formulated version of JIPDA. This comes down to pruning JIPDA permutation hypotheses in the sense of keeping the best permutation hypothesis only per combination of existing and detected targets and allocated measurements. In [11] it was shown that this kind of pruning results into an avoidance of track coalescence. Hence, the resulting filters are referred to as JPDA* and JIPDA*, where the * stand for "avoiding track coalescence".

Through initial Monte Carlo simulations with IPDA, JIPDA and JIPDA* on an illustrative example, the coalescence avoidance property of JIPDA* has been confirmed, and the potential benefits of using JIPDA* in difficult target crossing scenarios shown.

There are many directions in which the results of this paper can be extended. One of our priorities is to incorporate IMM for maneuvering target tracking with JIPDA*, using the descriptor system approach to incorporate IMM with JPDA* [15], [16], [17]. Another is to develop a better way of coordination between track initiation, confirmation, maintenance, merging and termination, e.g. [18].

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Apendix A (Proof of Proposition 1)

If $\phi = 0$ we get

$$p_{x_t|\xi_t,\phi_t,\tilde{\chi}_t,Y_t}(x \mid \xi, 0, \tilde{\chi}) = p_{x_t|\xi_t,Y_{t-1}}(x \mid \xi) \qquad \text{(A.1)}$$

Else, i.e. $\phi \neq 0$:

$$\begin{aligned}
&p_{x_t|\xi_t,\phi_t,\tilde{\chi}_t,Y_t}(x \mid \xi,\phi,\tilde{\chi}) = \\
&= p_{x_t|\xi_t,\phi_t,\tilde{\chi}_t,\mathrm{y}_t,L_t,Y_{t-1}}(x \mid \xi,\phi,\tilde{\chi},\mathrm{y}_t,L_t) = \\
&= p_{x_t|\xi_t,\phi_t,\tilde{\chi}_t,\mathrm{y}_t,L_t,\tilde{\mathrm{y}}_t,Y_{t-1}}(x \mid \xi,\phi,\tilde{\chi},\mathrm{y}_t,L_t,\underline{\tilde{\chi}}\mathrm{y}_t) = \\
&= p_{x_t|\xi_t,\phi_t,\tilde{\mathrm{y}}_t,Y_{t-1}}(x \mid \xi,\phi,\underline{\tilde{\chi}}\mathrm{y}_t) = \\
&= p_{\tilde{z}_t|x_t,\xi_t,\phi_t}(\underline{\tilde{\chi}}\mathrm{y}_t \mid x,\xi,\phi) \cdot p_{x_t|\xi_t,Y_{t-1}}(x \mid \xi)/F_t(\phi,\tilde{\chi},\xi)
\end{aligned} \tag{A.2}$$

with

$$F_t(\phi,\tilde{\chi},\xi) \stackrel{\triangle}{=} p_{\tilde{z}_t|\xi_t,\phi_t,Y_{t-1}}(\underline{\tilde{\chi}}\mathrm{y}_t \mid \xi,\phi) \tag{A.3}$$

(A.2) yields (18). Subsequently

$$\begin{aligned}
\beta_t(\xi,\phi,\tilde{\chi}) &\stackrel{\triangle}{=} \mathrm{Prob}\{\phi_t=\phi,\tilde{\chi}_t=\tilde{\chi},\xi_t=\xi \mid Y_t\} = \\
&= p_{\phi_t,\tilde{\chi}_t,\xi_t|Y_t}(\phi,\tilde{\chi},\xi) = \\
&= p_{\phi_t,\tilde{\chi}_t\xi_t|\mathrm{y}_t,L_t,Y_{t-1}}(\phi,\tilde{\chi},\xi \mid \mathrm{y}_t,L_t) = \\
&= p_{\mathrm{y}_t,\tilde{\chi}_t|\xi_t,\phi_t,L_t,Y_{t-1}}(\mathrm{y}_t,\tilde{\chi}|\xi,\phi,L_t) \cdot \\
&\quad \cdot p_{\phi_t,\xi_t|L_t,Y_{t-1}}(\phi,\xi \mid L_t)/c'_t = \\
&= p_{\mathrm{y}_t,\tilde{\chi}_t|\xi_t,\phi_t,L_t,Y_{t-1}}(\mathrm{y}_t,\tilde{\chi} \mid \xi,\phi,L_t) \cdot \\
&\quad \cdot p_{\phi_t|\xi_t,L_t,Y_{t-1}}(\phi \mid \xi,L_t) p_{\xi_t|Y_{t-1}}(\xi)/c'_t
\end{aligned} \tag{A.4}$$

If $\phi \neq 0$, we have $D_t > 0$ and

$$\tilde{\chi}_t^T\tilde{\chi}_t = \Phi(\psi_t)^T\chi_t\chi_t^T\Phi(\psi_t) = \Phi(\psi_t)^T\Phi(\psi_t) = \mathrm{Diag}\{\psi_t\} \tag{A.5}$$

Hence

$$\psi_t = \mathrm{Diag}\{\psi_t\}1_{L_t} = \tilde{\chi}_t^T\tilde{\chi}1_{L_t}$$

with 1_{L_t} an L_t column vector with L_t 1-valued components. Moreover, because

$$\tilde{\chi}_t\Phi(\psi_t)^T = \chi_t^T\Phi(\psi_t)\Phi(\psi_t)^T = \chi_t^T \tag{A.6}$$

we know that the transformation from (ψ_t,χ_t) into $\tilde{\chi}_t$ has an inverse. For the first term at the right hand side of (A.4) this implies:

$$\begin{aligned}
&p_{\mathrm{y}_t,\tilde{\chi}_t|\xi_t,\phi_t,L_t,Y_{t-1}}(\mathrm{y}_t,\chi^T\Phi(\psi) \mid \xi,\phi,L_t) = \\
&\quad = p_{\mathrm{y}_t,\psi_t,\chi_t|\xi_t,\phi_t,L_t,Y_{t-1}}(\mathrm{y}_t,\psi,\chi \mid \xi,\phi,L_t)
\end{aligned} \tag{A.7}$$

As the transformation from $(\mathrm{y}_t,\psi_t,\chi_t)$ into $(\tilde{z}_t,f_t,\psi_t,\chi_t)$ is a permutation, we get for $L_t > D(\phi) > 0$

$$\begin{aligned}
&p_{\mathrm{y}_t,\psi_t,\chi_t|\xi_t,\phi_t,L_t,Y_{t-1}}(\mathrm{y}_t,\psi,\chi \mid \xi,\phi,L_t) = \\
&= p_{\tilde{z}_t,f_t,\psi_t,\chi_t|\xi_t,\phi_t,L_t,Y_{t-1}}(\underline{\chi}^T\underline{\Phi}(\psi)\mathrm{y}_t,\underline{\Phi}(1_{L_t}-\psi)\mathrm{y}_t,\psi,\chi| \\
&\qquad |\xi,\phi,L_t)
\end{aligned} \tag{A.8}$$

Substituting (A.8) in (A.7) and this in (A.4) yields:

$$\begin{aligned}
&\beta_t(\phi,\chi^T\Phi(\psi),\xi) = \\
&p_{\tilde{z}_t,f_t,\psi_t,\chi_t|\xi_t,\phi_t,L_t,Y_{t-1}}(\underline{\chi}^T\underline{\Phi}(\psi)\mathrm{y}_t,\underline{\Phi}(1_{L_t}-\psi)\mathrm{y}_t,\psi,\chi| \\
&|\xi,\phi,L_t) \cdot p_{\phi_t|\xi_t,L_t,Y_{t-1}}(\phi \mid \xi,L_t) p_{\xi_t|Y_{t-1}}(\xi)/c'_t
\end{aligned} \tag{A.9}$$

Hence, for $L_t > D(\phi) > 0$, this yields:

$$\begin{aligned}
&\beta_t(\phi,\chi^T\Phi(\psi),\xi) = p_{\tilde{z}_t|\xi_t,\phi_t,Y_{t-1}}(\underline{\chi}^T\underline{\Phi}(\psi)\mathrm{y}_t|\xi,\phi) \cdot \\
&\quad \cdot p_{f_t|\phi_t,\psi_t,L_t}(\underline{\Phi}(1_{L_t}-\psi)\mathrm{y}_t \mid \phi,\psi) p_{\psi_t|\phi_t,L_t}(\psi \mid \phi) \cdot \\
&\quad \cdot p_{\chi_t|\phi_t}(\chi \mid \phi) p_{L_t|\phi_t}(L_t \mid \phi) p_{\phi_t|\xi_t}(\phi|\xi) p_{\xi_t|Y_{t-1}}(\xi)/c''
\end{aligned} \tag{A.10}$$

Evaluation of the terms in (A.10) yields:

$$\begin{aligned}
&p_{f_t|\phi_t,\psi_t,L_t}(\underline{\Phi}(1_{L_t}-\psi)\mathrm{y}_t|\phi,\psi) = \\
&= p_{f_t|F_t,\psi_t}(\underline{\Phi}(1_{L_t}-\psi)\mathrm{y}_t|L_t-D(\phi),\psi) = \\
&\stackrel{(6.b)}{=} \prod_{i=1}^{L_t-D(\phi)} p_f\big([\Phi(1_{L_t}-\psi)\mathrm{y}_t]_i\big) = \\
&= \prod_{i=1}^{L_t-D(\phi)} p_f\big([\Phi(1_{L_t}-\tilde{\chi}^T\tilde{\chi}1_{L_t})\mathrm{y}_t]_i\big)
\end{aligned} \tag{A.11}$$

$$p_{\psi_t|\phi_t,L_t}(\psi|\phi,L_t) = D(\phi)!(L_t-D(\phi))!/L_t! \tag{A.12}$$

$$p_{\chi_t|\phi_t}(\chi|\phi) = 1/D(\phi)! \tag{A.13}$$

$$\begin{aligned}
p_{L_t|\phi_t}(L_t|\phi) &= p_{F_t}(L_t-D(\phi)) = \\
&= (\hat{F}_t)^{(L_t-D(\phi))}\exp\{-\hat{F}_t\}/(L_t-D(\phi))! \\
&\qquad \text{if } L_t \geq D(\phi) \\
&= 0 \qquad \text{if } L_t < D(\phi)
\end{aligned} \tag{A.14}$$

$$p_{\phi_t|\xi_t}(\phi|\xi) = \prod_{i=1}^{M}\left[(\xi^i P_{\mathrm{d}}^i)^{\phi_i}(1-\xi^i P_{\mathrm{d}}^i)^{1-\phi_i}\right] \tag{A.15}$$

with the last equation following from (5.b).
Substituting (A.3) and (A.11) through (A.15) into (A.10) and subsequent evaluation yields for $L_t > D(\phi) > 0$:

$$\begin{aligned}
&\beta_t(\phi,\chi^T\Phi(\psi),\xi) = F_t(\phi,\chi^T\Phi(\psi),\xi) \cdot \\
&\quad \cdot \hat{F}_t^{(L_t-D(\phi))} \cdot \prod_{j=1}^{L_t-D(\phi)} p_f\big([\Phi(1_{L_t}-\tilde{\chi}^T\tilde{\chi}1_{L_t})\mathrm{y}_t]_j\big) \cdot \\
&\quad \cdot \prod_{i=1}^{M}[(\xi^i P_d^i)^{\phi_i}(1-\xi^i P_d^i)^{(1-\phi_i)}] \cdot p_{\xi_t|Y_{t-1}}(\xi)/c_t
\end{aligned}$$

with c_t a normalizing constant. It can be easily verified that the last equation also holds true if $L_t = D(\phi)$ or if $D(\phi) = 0$. Together with (6.c) this yields (19). □